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ON TRIGONOMETRIC POLYNOMIALS WITH COEFFICIENTS $+1, -1, 0$

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1°. *Notation.* In what follows Λ denotes a finite collection of natural numbers, $|\Lambda|$ denotes the number of elements in Λ, and $\deg\Lambda = \max\{k : k \in \Lambda\}$. For a given sequence $z = \{z_k\}_1^\infty$ we set

$$\operatorname{supp} z = \{k : z_k \neq 0\}, \qquad \|z\|_{l_2(\Lambda)} = \left(\sum_{k\in\Lambda} z_k^2\right)^{1/2}.$$

Finally, let $A(\Lambda) = \{z = \{z_k\}_1^\infty\colon\ z_k = \pm 1, -1, \text{ or } 0, \operatorname{supp} z \subset \Lambda\}$.

2°. The Rudin-Shapiro polynomials are known in the theory of trigonometric series (cf. [1], Chapter 4, §4, Theorem 11):

$$P_N(x) = \sum_{k=1}^{N} \varepsilon_k e^{ikx}, \qquad \varepsilon_k = \pm 1,$$
$$\|P_N\|_{C(-\pi,\pi)} \leq 5N^{1/2}, \qquad N = 1, 2, \dots.$$

In this note we consider the question of constructing for a given set Λ polynomials of the form

$$P_\Lambda = P_\Lambda(x) = \sum_{k\in\Lambda} \varepsilon_k e^{ikx}, \qquad \varepsilon_k = \pm 1,\ x \in (-\pi, \pi), \tag{1}$$

with the smallest possible maximum modulus. The method of constructing the Rudin-Shapiro polynomials is very special and inapplicable to the present problem. The well-known probabilistic method (cf. [1], Chapter 4, §4, Theorem 10) gives only the existence of a polynomial of the form (1) with $\|P_\Lambda\|_{C(-\pi,\pi)} \leq C(1 + |\Lambda| \ln \deg \Lambda)^{1/2}$.

Here, using geometric estimates, we establish

PROPOSITION 1. *For each $\delta > 0$ and each set Λ, $\deg\Lambda \leq N$, $|\Lambda| \geq \delta N$:*
a) *there exists a polynomial P_Λ of the form* (1) *with*

$$\|P_\Lambda\|_{C(-\pi,\pi)} \leq C_\delta |\Lambda|^{1/2} \ln\ln(|\Lambda| + 3);$$

b) *there exists a polynomial $Q_\Lambda(x) = \sum z_k e^{ikx}$, $\{z_k\} \in A(\Lambda)$ with $\sum_{k\in\Lambda} |z_k| \geq |\Lambda|/2$ and*

$$\|Q_\Lambda\|_{C(-\pi,\pi)} \leq C_\delta |\Lambda|^{1/2}$$

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(*we denote by* C_δ *constants depending only on the variable* δ).

PROPOSITION 2. *For any collection of vectors* $\{e_j\}_1^m \subset l_2^n$ *with* $\|e_j\|_{l_2^n} = 1$, $\|e_j\|_{l_\infty^n} \le Bn^{-1/2}$, $j = 1, \dots, m$, *there exists a vector* $\varepsilon = \{\varepsilon_k\}_1^n$, $\varepsilon_k = \pm 1$, *such that*

$$|(\varepsilon, e_j)| \le C_B(m/n)^{1/2} \ln\ln(m+3), \qquad j = 1, \dots, m$$

(*here* (x, y) *denotes the inner product in the n-dimensional Euclidean space* l_2^n).

Point a) of Proposition 1 is easy to deduce from Proposition 2 if we take account of the fact that $\|P\|_{C(-\pi,\pi)} \asymp \max_{1\le k\le 10N} |P(2\pi k/10N)|$ for each trigonometric polynomial of degree $\le N$. Proposition 2 in turn and point b) of Proposition 1 are deduced from the following result, obtained in [2].

PROPOSITION 3. *For any collection of vectors* $\{e_j\}_1^m \subset l_2^n$ *with* $\|e_j\|_{l_2^n} = 1$, $j = 1, \dots, m$, *there exists a vector* $z = \{z_k\}_1^n$, $z_k = \pm 1, -1$, *or* 0, *with* $|\operatorname{supp} z| \ge n/6$ *and such that*

$$|(z, e_j)| \le K(m/n)^{1/2}, \qquad j = 1, \dots, m,$$

where K *is an absolute constant.*

We explain the deduction of Proposition 2. Suppose the collection $\{e_j\}$ satisfies the hypotheses of this proposition. Then for any $\Lambda \subset \{1, \dots, n\}$

$$\|e_j\|_{l_2(\Lambda)} \le Bn^{-1/2}|\Lambda|^{1/2}, \qquad j = 1, \dots, m. \tag{2}$$

Applying Proposition 3, we find a vector $z^{(1)} \subset A(\Lambda_1)$, $\Lambda_1 \equiv \{1, \dots, n\}$, $|\operatorname{supp} z^{(1)}| \ge n/6$, for which

$$|(z^{(1)}, e_j)| \le K(m/n)^{1/2}, \qquad j = 1, \dots, m.$$

Set $\Lambda_2 = \Lambda_1 \setminus \operatorname{supp} z^{(1)}$. Then, again applying Proposition 3 (cf. also (2)), we find a vector $z^{(2)} \in A(\Lambda_2)$, $|\operatorname{supp} z^{(2)}| \ge |\Lambda_2|/6$, for which

$$|(z^{(2)}, e_j)| \le K(m/n)^{1/2} |e_j|_{l_2(\Lambda)} \le KB(m/n)^{1/2}, \qquad j = 1, \dots, m.$$

Set $\Lambda_3 = \Lambda_2 \setminus \operatorname{supp} z^{(2)}$. Continuing the process and using Proposition 3 at each step, we construct vectors $z^{(1)}, \dots, z^{(s)}$, $s \le 10 \ln\ln(m+3)$ such that $z^{(q)} \in A(\Lambda_q)$, $\operatorname{supp} z^{(q)} \cap \operatorname{supp} z^{(q')} = \varnothing$ if $q \ne q'$, $1 \le q, q' \le s$, and

$$|\Lambda_{s+1}| \equiv \left|\Lambda_1 \Big\backslash \bigcup_{1\le q\le s} \operatorname{supp} z^{(q)}\right| \le n(5/6)^s < n/\ln m,$$

$$|(z^{(q)}, e_j)| \le KB(m/n)^{1/2}, \qquad q = 1, \dots, s, \ j = 1, \dots, m.$$

Then from probabilistic considerations we find a vector $\varepsilon' = \{\varepsilon_k'\}_1^n$, $\varepsilon_k' = \pm 1$ for $k \in \Lambda_{s+1}$ and $\varepsilon_k' = 0$ for $k \notin \Lambda_{s+1}$, such that (cf. (2))

$$|(\varepsilon', e_j)| \le C(1 + \ln m)^{1/2} \|e_j\|_{l_2(\Lambda_{s+1})} \le C, \qquad j = 1, \dots, m.$$

Then the vector $\varepsilon = \varepsilon' + \sum_{q=1}^s z^{(q)}$ satisfies the requirements of Proposition 2.

In conclusion we note that analogues of Proposition 1 hold also for polynomials in other systems of functions, in particular for trigonometric polynomials in several variables (cf. also [3]) and polynomials in the Walsh system.

Bibliography

1. B. S. Kashin and A. A. Saakyan, *Orthogonal series*, "Nauka", Moscow, 1984; English transl., Amer. Math. Soc., Providence, R.I., 1989.

2. B. S. Kashin, *An isometric operator in* $L^2(0,1)$, C. R. Acad. Bulgare Sci. **38** (1985), 1613–1616. (Russian)

3. A. F. Koregin, *On absolute convergence of multiple trigonometric series with gaps*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1985**, no. 4, 3–9; English Transl. in Moscow. Univ. Math. Bull. **40** (1985).

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