ON ESTIMATES OF DIAMETERS

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This paper contains highlights of the report delivered by the author at the conference.

If K is a compact set in a Banach space X, then the quantity

(1) $$d_n(K,X) = \inf_{L,\ \dim L \leq n} \Delta_X(K,L)$$

is called the n-th diameter of this set in the sense of Kolmogorov; in the above definition

(2) $$\Delta_X(K,L) = \sup_{y \in K} \inf_{z \in L} \|y-z\|_X .$$

Starting in the fifties, dozens of papers have been dedicated to the problem of estimating diameters (c.f. [11], [5]). In recent years, estimates of diameters have found applications in the qualitative analysis of various computational algorithms as well as in functional analysis and in the investigation of distribution of the eigenvalues of various operators. In this context, the problem arises to estimate the diameters of certain compact sets of functions in the L^q-metric, $1 \leq q \leq \infty$ ($L^\infty = C$).

A number of papers by various authors are dedicated to the best-known problem of this kind, namely to the problem of estimating the quantities

(3) $$d_n(W_p^r, L^q(0,1)),\ 1 \leq p,\ q \leq \infty,\ r > 0$$

where W_p^r stands for the class of smooth functions f(x) which possess r-1 absolutely continuous derivatives and $\|f\|_{L^p(0,1)} + \|f^{(r)}\|_{L^p(0,1)} \leq 1$; here, for fractional r, the

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derivative $f^{(r)}$ is understood in the generalized sense. The quantity (3) is interesting only when W_p^r is compact in L^q, i.e. if $r > \frac{1}{p} - \frac{1}{q}$.

An investigation of the orders of magnitude of the quantity (3) as $r > 1$, $n \to \infty$ was completed by the author in 1976. The exact result has the following form: If $1 \leq p$, $q \leq \infty$, $rp > 1$, then

$$(4) \quad d_n(W_p^r, L^q) \sim \begin{cases} n^{-r} \text{ if } p \geq q \text{ or } 2 < p < q \\ n^{-r-\frac{1}{2}+\frac{1}{p}} \text{ if } p \leq 2 < q \\ n^{-r-\frac{1}{q}+\frac{1}{p}} \text{ if } 1 \leq p < q \leq 2 . \end{cases}$$

Relation (4) has been proved in the following way: In the case $q \leq \max(p,2)$, it was proved by the joint efforts of several authors (for details c.f. [11]) that

$$(5) \quad d_n(W_p^r, L^q) \asymp \Delta_{L^q}\left(W_p^r, T^{[n/2]}\right),$$

where

$$T^n = \left\{P(t): \quad P(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)\right\}.$$

The estimate

$$\Delta_{L^q}(W_p^r, T^n) \sim \begin{cases} n^{-r+\frac{1}{p}-\frac{1}{q}} \text{ if } p < q \\ n^{-r} \text{ if } p \geq q \end{cases}$$

has been known since the thirties and thus the estimate for $d_n(W_p^r, L^q)$ from above was not difficult in this case. For estimates of the diameter (3) from below in the case $p \geq q$, topological considerations were applied by Tihomirov, and later by other authors, which gave not only the proof of these estimates from below but also the computation of the exact value of (3) in several cases. For $1 \leq p < q \leq 2$, the estimates from below may be obtained fairly simply. The starting point here was obtained in the beginning of the fifties by W. Rudin and S.B. Stechkin, who proved (4) for $p=1$, $q=2$. The complete case $1 \leq p < q \leq 2$ is contained in [2].

It turned out that the behavior of the diameters (3) in the case $q > \max(p,2)$, is entirely different from that in the first case. In 1974, R. S. Ismagilov proved that

$$d_n(W_1^2,C) \leqq C_\varepsilon n^{-\frac{6}{5}+\varepsilon} \leqq Cn^{-1} \leqq \Delta_C(W_1^2,T^n);\ n > n_0.$$

This result implies (c.f. E. D. Gluskin [1]) that already in the case $p = 1$, $q = \infty$, $r = 2$ the relation (5) is no longer true and, consequently, the trigonometric system is not a good approximating subspace for the class W_1^2 in the uniform norm. The same deficiency is possessed by the spaces of splines and algebraic polynomials. It was shown by the author that, in general, for $q > \max(p,2)$, spaces, which are in a certain sense randomly chosen n-dimensional subspaces, are close to the best and, on that basis, the orders of the diameters (3) were computed.

Usually, for estimates of the diameter (3), a discretization of the problem was used, namely the reduction of it to the estimate of the diameters

(6) $\quad d_n(B_p^m,\ell_q^m)$,

where $B_p^m = \{x \in R^m:\ \|x\|_{\ell_p^m} \leqq 1\}$ and, for $x = \{x_i\} \in R^m$

$$\|x\|_{\ell_q^m} = \begin{cases} \left(\sum_1^m |x_i|^q\right)^{1/q}, & 1 \leqq q < \infty \\ \max|x_i|, & q = \infty. \end{cases}$$

That is, the asymptotic behavior of $d_n(W_1^r,L^2)$ has been determined in [10] by using the result $d_n(B_1^m,\ell_2^m) = (\frac{m-n}{n})^{1/2}$. A sufficiently sharp method of discretization is contained in [1] and in [8]. The main difficulty in the estimation of the diameters (3) consists in the sharp evaluation of (6). Note that with the aid of estimates of the diameters (6) it is possible to estimate the diameters of other classes of functions with that or other degrees of accuracy.

In case $q \geqq \max(p,2)$, routine subspaces could not supply satisfactory estimates for (6). In 1974, a probabilistic approach was proposed by the author; this approach may be briefly described as follows. A sufficiently rich family, $\Omega_{m,n}$, of n-dimensional subspaces L in R^m, is considered endowed with some measure μ (for example, the set of all n-dimensional subspaces in R^m with Haar measure or the set of subspaces $L \subset R^m$, $\dim L = n$, which are described, using the standard basis, by the matrix with elements ± 1, and $\mu L = 2^{-mn}$); the quantity

$$I_{p,q} = \int_{\Omega_{m,n}} \Delta_{\ell_q}(B_p^m, L)\,d\mu$$

is evaluated.

It is clear that

$$(7) \quad d_n(B_p^m, \ell_q^m) = \inf_{L,\, \dim L = n} \Delta_{\ell_q}(B_p^m, L) \leqq I_{p,q}.$$

It turns out that the difference between the left- and right-hand sides in (7) is "not large", and thus if we get a sharp estimate of the integral $I_{p,q}$, we get a good estimate for the diameter (6), as well. For example, if we take as $\Omega_{m,n}$ the set of all n-dimensional subspaces, then it may be proved that for arbitrary p and q, $1 \leqq p,\ q \leqq \infty$, as $n \to \infty$, the estimate

$$d_n(B_p^{2n}, \ell_q^{2n}) \sim I_{p,q} \sim \begin{cases} 1 \text{ if } 1 \leqq p < q \leqq 2 \\ n^{-\frac{1}{2}+\frac{1}{q}} \text{ if } p < 2 < q \\ n^{-\frac{1}{p}+\frac{1}{q}} \text{ if } p \geqq \min(2,q) \end{cases}$$

is valid.

Thus, random n-dimensional subspaces provide a good approximation for the ball B_p^{2n} in the ℓ_q^{2n} metric for all p and q, while the n-dimensional space of discrete trigonometric polynomials provides a good approximation for B_p^{2n} only if $q \leqq \max(2,p)$. Using this method, the author [5] proved the following:

THEOREM 1. If $m > n$, then

$$\frac{1}{2}n^{-1/2} < d_n(B_2^m, \ell_\infty^m) < Cn^{-1/2}(1+\ell n(m/n))^{3/2}.$$

Estimates of the diameters $d_n(B_2^m, \ell_\infty^m)$ turn out to be the most important for applications; thus Theorem 1 was efficient enough to compute the orders of magnitude of the diameters (3) for all p and q.

A special feature of the behavior of the diameters (6) is their weak dependence upon m. The latter fact may be seen from Theorem 1 and is still more clear from the results in [4].

THEOREM 2. If $\gamma n < m < n^\lambda$, $\gamma > 1$, $\lambda > 1$, then

$$C_\gamma n^{-1/2} < d_n(B_1^m, \ell_\infty^m) < C_\lambda n^{-1/2}.$$

Another peculiarity is the fact that from the estimates of $d_n(B_p^m, \ell_\infty^m)$, sufficiently sharp estimates of $d_n(B_p^m, \ell_q^m)$ for $q < \infty$ follow in a trivial way. For example, the inequality $\|x\|_{\ell_q^m} \leq m^{1/q}\|x\|_{\ell_\infty^m}$ and Theorem 2 imply that for $q > 2$

$$d_n(B_1^m, \ell_q^m) \leq C_q \min(1, m^{1/q}\, n^{-1/2}).$$

On the other hand, the estimate of $d_n(B_1^m, \ell_q^m)$ from below has the same form. More precisely, the following statement holds:

THEOREM 3. If $m \geq 2n$ and $q > 2$, then

$$\frac{1}{4}\min(1, m^{1/q}\, n^{-1/2}) \leq d_n(B_1^m, \ell_q^m) \leq C_q \min(1, m^{1/q}\, n^{-1/2}).$$

It follows that the general behavior of the diameters (6) up to logarithm factors is now clear, but exact estimates uniform in m and n are still not known in the most interesting case, $q = \infty$.

The following problems are of special interest to the author:

1. Determine the exact order (as $n \to \infty$) of the quantity $d_n\left(B_2^{n^2}, \ell_\infty^{n^2}\right)$.

2. Determine, for all m and n, the order of magnitude of $d_n(B_1^m, \ell_\infty^m)$.

It should be noted in connection with problem 2 that $d_n(B_1^m, \ell_\infty^m) \leqq Cn^{-1/2}(1+\ell n(m/n))^{1/2}$ for all m and n (see [5]) and that the exponent 1/2 in the factor $(1+\ell n(m/n))$ cannot be decreased since $d_n\left(B_1^{K^n}, \ell_\infty^{K^n}\right) \geqq 1/4$ for K sufficiently large (see [9]).

It is known that the following quantity is in a certain sense dual to the diameter (1):

$$(8) \quad d^n(K,X) = \inf_{L,\ \operatorname{codim} L = n} \ \sup_{y \in L \cap K} \|y\|_X$$

For example, the following relation holds:

$$d_n(B_p^m, \ell_q^m) = d^n(B_{q'}^m, \ell_{p'}^m), \ \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1.$$

Here, we present the following simple result which provides an estimate from below of the quantity (8) in the case $X = \ell_2^n$ and which may be applied to a large variety of sets K.

THEOREM 4 (c.f. [6]). For any central-symmetric and convex set $K \subset B_2^m$, and any integer r, $1 \leqq r \leqq m$, we have

$$\inf_{L,\ \dim L \geq r} \ \sup_{y \in L \cap K} \|y\|_{\ell_2^m} \geq \left[V(K)(V_r V_{m-r})^{-1}\right]^{1/r},$$

where $V(K)$ denotes volume of the set K, and $V_s = \pi^{s/2}\Gamma^{-1}(\frac{s}{2}+1)$ is volume of the ball B_2^s.

The following result provides a lower bound for diameters in ℓ_2^n of sets which lie on the Euclidean sphere in R^n.

THEOREM 5 (c.f. [7]). *Given* $\varepsilon > 0$, *there is a number* $\delta(\varepsilon) > 0$ *such that for any* $G \subset \{x : \|x\|_{\ell_2^n} = 1\}$ $1 \le n \le \infty$ *and any array* $\{i_r\}_{r=1}^{n-j}$, $1 \le i_1 < \ldots < i_{n-j} \le n$ *and* $j \le n\delta(\varepsilon)$, *with the property* $G \cap \{x \varepsilon R^n : x_{i_1} = \ldots = x_{i_{n-j}} = 0\} \ne \phi$, *we have*

$$1 - \varepsilon \le d_j(G, \ell_2^n) \le 1.$$

Theorem 5 is a consequence of the next result. Given a matrix $A = (a_{ij})$, $1 \le i \le i'$, $1 \le j \le j'$, let

$$\|A\| = \sup_{\sum_{i=1}^{i'} x_i^2 = 1 = \sum_{j=1}^{j'} y_j^2} \sum_{i=1}^{i'} \sum_{j=1}^{j'} a_{ij} x_i y_j .$$

THEOREM 6. *If* $A = (a_{ij})$, $1 \le i \le m$, $1 \le j \le n$ *and* $m > n$, *then there is a set* $\Omega = \{i_\nu\}_{\nu=1}^n$ *such that*

$$\|(a_{ij}),\ i \varepsilon \Omega,\ 1 \le j \le n\| \le C\, \ell n^{-1/2}(m/n) \|A\|$$

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