ON A PROPERTY OF FUNCTIONAL SERIES

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The question of the convergence of functional series everywhere in the segment [0, 1] is considered. Let $F = \{f\}$ be the set of such functions in [0, 1] for each of which there is a transposition of the series $\sum_{k=1}^{\infty} f_k(x)$, which converges to it everywhere in [0, 1]. An example of a series is constructed such that the set $F$ consists just of an identical zero, but $\sum_{k=1}^{\infty} |f_k(x)| = \infty$ for any point of the segment [0, 1].

Let be given the number series

$$\sum_{n=1}^{\infty} a_n. \quad (1)$$

Let $A$ denote the set of all those numbers $a$ for each of which there is a transposition $\tau = \{n_k\} = (n_1, n_2, \ldots, n_k, \ldots)$ of the natural number series such that

$$a = \sum_{k=1}^{\infty} a_{n_k} = \sum_{i} a_i. \quad \text{(2)}$$

It is well known that if the set $A$ consists of one point, then the series (1) is absolutely convergent. This results directly from the Riemann theorem. This fact also holds for series of the form (1) with elements $a_n$ from $N$-space [Steinitz-Levy theorem (see [1, 2])].

An analogous question is investigated herein for the case when $a_n = a_n(x)$ are continuous functions in the segment [0, 1]. More accurately, let the functions $f_n(x) \in C(0, 1)$ and $F = \{f(x)\}$ be a set of all those functions $f(x)$ defined in [0, 1] for each of which there is a transposition $\tau = \{n_k\}$ such that the series

$$\sum_{k=1}^{\infty} f_{n_k}(x) = \sum_{i} f_i(x) \quad \text{(3)}$$

converges at each point $x \in [0, 1]$ to $f(x)$.

Could it be asserted that if the set $F = \{f(x)\}$ consists of just one function then the series (2) will converge absolutely to some points of the segment [0, 1]? The answer to this question is given by the following

**THEOREM.** There exists a series

$$\sum_{n=1}^{\infty} f_n(x) \quad (x \in [0, 1]; \quad f_n(x) \in C(0, 1); \quad n = 1, 2, \ldots) \quad \text{(4)}$$

such that the set $F$ consists of one function $f(x) \equiv 0$ and at the same time

$$\sum_{n=1}^{\infty} |f_n(x)| = \infty \quad \text{for all} \quad x \in [0, 1]. \quad \text{(5)}$$


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Proof. Let $A$ be the set of all transpositions $\tau = \{n_k\}$ of the natural number series such that
\[ n_k \leq 27k^3 \quad \text{for} \quad k = 1, 2, \ldots. \tag{5} \]
Let us map the set $A$ in the segment $[0, 1]$ according to the rule
\[ x = \varphi(\tau) = 0, \frac{01}{n_1} \ldots \frac{01}{n_2} \ldots \frac{01}{n_3} \ldots \frac{01}{n_4} \ldots, \]
where the decimal fraction written out yields the number $x \in [0, 1]$. Let $B = \varphi(A)$ and $\overline{B}$ be the closure of the set $B$. It is easy to see that if $x \in \overline{B}$, then
\[ x = 0, 00 \ldots 011 \ldots 100 \ldots 011 \ldots 1 \ldots, \tag{6} \]
where $p_i \leq 27i^3$ ($i = 1, 2, \ldots$) and $p_j \neq p_k$ for $j \neq k$. Let us note that the sequence $\{p_i\}$ is not generally a transposition of the natural number series since some natural numbers may be missing in the set of numbers $p_i$ ($i = 1, 2, \ldots$).

Let us define the sequence of functions $\{\varphi_n(x)\}_{n=1}^{\infty}$ in the set $\overline{B}$. Let $x \in \overline{B}$; then $x$ is represented as the decimal fraction (6). Let us put
\[ \varphi_{n+1}(x) = \frac{1}{3^n} \quad \varphi_n(x) = -\frac{1}{3^n} \quad (j = 1, 2, \ldots). \tag{7} \]
It is clear that the function $\varphi_n(x)$ retains its sign in the set $\overline{B}$. Moreover, the functions $\varphi_n(x)$ are continuous in the set $\overline{B}$ for any $n = 1, 2, \ldots$. This results from the fact that if $j$ is a fixed integer, and $x_0 \in \overline{B}$, then there is a $\delta > 0$ such that the functions $\varphi_{2j}(x)$ and $\varphi_{2j+1}(x)$ are constant in the set $\overline{B} \cap (x_0 - \delta, x_0 + \delta)$.

Let us note first that for any point $x \in \overline{B}$ we will have [see (5)-(7)]
\[ \sum_{n=1}^{\infty} |\varphi_n(x)| = \infty \quad \text{for} \quad x \in \overline{B}, \tag{8} \]
and hence
\[ \sum_{n=1}^{\infty} \varphi_n(x) = 0 \quad \text{for} \quad x \in \overline{B}. \tag{9} \]
Furthermore, $\sum_{n=1}^{2k} \varphi_n(x) = 0$ for $x \in \overline{B}$ and $k = 1, 2, \ldots$, but $\lim_{n \to \infty} \varphi_n(x) = 0$ for $x \in \overline{B}$ [see (7)]. Hence, the series
\[ \sum_{n=1}^{\infty} \varphi_n(x) \tag{10} \]
converges in the set $\overline{B}$ to the function $\varphi(x) = 0$. Let us note that besides the functions $\varphi_n(x)$, the function $\varphi(x)$ which is unique for each $\varphi_n$ is also a member of the series (9).

We now prove that if the series (9) converges after some transposition $\tau' = \{m_i\}$ to some function $\Phi(x)$ everywhere in $\overline{B}$, then $\Phi(x) = \varphi(x) = 0$ for $x \in \overline{B}$, i.e.,
\[ \Phi(x) = \sum_{i=1}^{\infty} \varphi_{m_i}(x) = \sum_{i=1}^{\infty} \varphi_n(x) = 0. \tag{10} \]

Let us take an arbitrary natural number $N$ and let us call the function $\varphi_{j}(x)$ contracted into the section from 1 to $N$ in the transposition $\tau'$, if among the functions $\{\varphi_{m_1}(x)\}_{i=1}^{N}$ there is both the function $\varphi_{j}(x)$ and the function $\varphi_{j+1}(x)$. Let $K(\tau', N)$ denote the number of functions not reduced in the transposition $\tau'$ in the section from 1 to $N$.

Let us consider the series
\[ \sum_{i=1}^{\infty} \varphi_{m_i}(x_0) = \sum_{i} \varphi_{n}(x_0) \tag{11} \]
at an arbitrary point \( x_0 \in \overline{B} \). Let us call the number \( a_j = \varphi_{m_j}(x_0) \) reduced in the section from 1 to N in the cross \( \tau' \) if among the numbers \( \{a_{m_j}\}_{j=1}^{N} \) there are both the numbers \( a_j \) and \(-a_j\). Let \( K(x_0, \tau', N) \) denote the number of terms in the series (11) not reduced in the section from 1 to N.

Let us note that when the point \( x_0 \in \overline{B} \) [see (6)], then \( p_i \neq p_j \) for \( i \neq j \), and hence, if \( \varphi_n(x_0) = \varphi_m(x_0) \), then \( n = m \). Consequently, for any \( x_0 \in \overline{B}, \tau', \) and N the function \( K(x_0, \tau', N) = K(\tau', N) \), and the functions \( \varphi_{m_j}(x) \) not being reduced on the section from 1 to N in the series (10), are at the same places as the terms \( \{a_{m_j}\} \) of the series (11) not being reduced in the same section.

For a given transposition \( \tau' = \{m_j\} \) [see (10)] two cases are possible:

I. \( \lim_{N \to \infty} K(\tau', N) < \infty \)

II. \( \lim_{N \to \infty} K(\tau', N) = \infty \).

Members of the series (11) tend to zero, and moreover, for each member \( \varphi_{m_j}(x_0) \) of the series (11) there is a member \( \varphi_{m_j}(x_0) = -\varphi_{m_j}(x_0) \) of the same series. Hence, in case I, i.e., when \( K(\tau', N) \leq M = \text{const} \) for all \( N = 1, 2, \ldots \), in the partial sum

\[
\sigma_N(x_0) = \sum_{i=1}^{N} \varphi_{m_j}(x_0)
\]

the terms not being reduced (\( \leq M \) in quantity) have an arbitrarily large number as \( N \to \infty \). This means that \( \sigma_N(x_0) \to 0 \) as \( N \to \infty \). Therefore, the series (11) converges to zero at each point \( x_0 \in \overline{B} \), i.e., \( \Phi(x) = \varphi(x) \) and (10) is valid in case I.

Now, let us analyze case II. We prove that it is impossible. For this case we construct the divergent series

\[
\sum_{i=1}^\infty s_i, \quad (12)
\]

which is a transposition of the series

\[
\sum_{i=1}^\infty \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) \quad (13)
\]

and for which \( s_i = \varphi_{m_j}(x_i) \) (\( i = 1, 2, \ldots \)) at some point \( x_i \in \overline{B} \). This fact will contradict the convergence of the series (10) to the function \( \Phi(x) \) in the set \( \overline{B} \). Let us select two partial series from the series (13)

\[
T_1 = \sum_{k=1}^\infty \left( \frac{1}{\sqrt{2k}} - \frac{1}{\sqrt{2k+1}} \right), \quad T_2 = \sum_{k=1}^\infty \left( \frac{1}{\sqrt{2k-1}} - \frac{1}{\sqrt{2k+1}} \right).
\]

The remaining terms in (13) form some partial series which we denote by

\[
T_3 = \sum_{k=1}^\infty \left( \frac{1}{\sqrt{2k} - \sqrt{2k+1}} \right) \quad (14)
\]

It is clear that \( r_i \geq 3i \) (\( i \geq 1 \)). We construct the series (12) by induction.

**First Step in the Induction.** Let us select a natural number \( p_1 \) such that

\[
\sum_{k=1}^{p_1} \frac{1}{\sqrt{2k}} > 2 \sqrt{p_1}. \quad (15)
\]

After this, let us find the natural number \( N_1 \) such that

\[
K(\tau', N_1) > 2p_1 \quad (16)
\]

and the partial sum of the series (10)

\[
\sigma_{N_1}(x) = \sum_{i=1}^{N_1} \varphi_{m_j}(x) \quad (17)
\]
contains all functions \{\varphi_n(x)\}_{n=1}^{2p_1} (which are naturally reduced). Let \(\alpha^+(\tau', N)\) [or \(\alpha^- (\tau', N)\)] denote the number of positive (or negative) terms from 1 to N not reduced in the series (10). By virtue of (16), either \(\alpha_1 = \alpha^+(\tau', N) > p_1\), or \(\alpha_1 = \alpha^- (\tau', N) > p_1\). Without limiting the generality (this will be seen from the reasoning below), we can consider that

\[\alpha_1 > p_1.\]  

(18)

Two cases are possible: a) \(\beta_1 < p_1\), b) \(\beta_1 \geq p_1\). In case a) we set terms of the series \(T_1\) successively in the first \(p_1\) places of the positive functions not being reduced in (17):

\[\frac{1}{\sqrt{2 \cdot 1}}, \frac{1}{\sqrt{2 \cdot 2}}, \ldots, \frac{1}{\sqrt{2 \cdot p_1}},\]

i.e., if \(\varphi_{m_{1q}}(x) (1 < i_2 < \ldots < i_{\alpha_1})\) are positive functions not being reduced in the partial sum (17), then, [see (18)]

\[s_{m_{1q}} = \frac{1}{\sqrt{2 \cdot q}} \quad \text{for} \quad 1 \leq q \leq p_1.\]

Analogously, we successively set members of the series \(T_2\) in the \(\beta_1\) places of the negative functions not reduced in (17):

\[-\frac{1}{2 \cdot 1 + 1}, -\frac{1}{2 \cdot 2 + 1}, \ldots, -\frac{1}{2 \cdot \beta_1 + 1}.\]

Furthermore, we set terms from the series \(T_3\) successively at the places of the functions \(\varphi_{m_{1q}}(x)\) with \(p_1 < q < 2p_1 - \beta_1:\)

\[\frac{1}{2(\beta_1 + 4) + 1}, \ldots, \frac{1}{2p_1 + 1}.\]

Therefore, we have determined \(s_i\) for some values of \(i (1 \leq i \leq N_1)\) within the quantity \(p_1 + \beta_1 + (p_1 - \beta_1) = 2p_1\). We set members of the series \(T_3\) at the places of the functions \(\varphi_{m_{1q}}(x)\) with \(2p_1 - \beta_1 < q \leq \alpha_1\) and at the places of all the functions reduced in (16) so that:

1) If there are functions \(\varphi_{2j-4}(x)\) and \(-\varphi_{2j-4}(x) = \varphi_{2j}(x)\) at any two places in (17), then we put \((1/\sqrt{\tau})\) and \(-1/(\sqrt{\tau})\), respectively, at these places;

2) At the place of the functions \(\varphi_{m_{1q}}(x) = \varphi_{2j-1}(x)\) with \(2p_1 - \beta_1 < q < \alpha_1\), we put \((1/\sqrt{\tau})\);

3) If the first member of the series \(T_3\) unused in 1) and 2) has the number \(i_0 \leq p_1\), then at the place of the function \(\varphi_{m_{\alpha_1}}(x) = \varphi_{2j-1}(x)\), we put \((1/\sqrt{\tau})\). If \(i_0 > p_1\), then at the place of \(\varphi_{m_{\alpha_1}}(x) = \varphi_{2j-1}(x)\), we put \(1/(\sqrt{\tau})\) as in 2).

We have therefore constructed \(s_i\) for all \(1 \leq i \leq N_1\). The numbers \(s_i (1 \leq i \leq N_1)\) have been constructed so that if the number \(s_i = (1/\sqrt{\tau})\) [or \(-1/(\sqrt{\tau})\)] has been compared to the functions \(\varphi_{m_1}(x) = \varphi_{2h-1}(x)\) or \(\varphi_{m_2}(x) = \varphi_{2h}(x)\), then [see (14) and the choice of \(N_1\)]

\[l \leq 27 \cdot h^3.\]  

(19)

Let us note that [see (15)]

\[A_1 = \sum_{i=1}^{N_1} s_i > \sum_{i=1}^{N_1} \frac{1}{\sqrt{2j}} - \sum_{i=1}^{N_1} \frac{1}{\sqrt{2j+4}} = R_1 > \sqrt{p_1}.\]  

(20)

In case b), i.e., when \(\beta_1 \geq p_1\), we construct \(s_i\) as follows. We put the number \((1/\sqrt{2 \cdot q})\) successively for \(1 \leq q \leq p_1\) [or \(-1/(2q + 1)\) for \(1 \leq q \leq p_1\)] at the first \(p_1\) places of the positive (or negative) functions not reduced in (17).

We put members of the series \(T_3\) at the remaining places in (17) exactly as in sections 1), 2), and 3) case a).
Let $s^{(1)}_i$ denote the numbers thus constructed for $1 \leq i \leq N_1$. If we alter the construction just so that we put successively the number $1/(2q+1)$ with $1 \leq q \leq p_1$ (or $-1/(2q+1)$ with $1 \leq q \leq p_1$) in the first $p_1$ places of the positive (or negative) functions not reduced in (17), then we obtain the number $s^{(2)}_i$ with $1 \leq i \leq N_1$. But then, [see (20)]

$$\left| \sum_{i=1}^{N_1} s^{(2)}_i - \sum_{i=1}^{N_1} s^{(1)}_i \right| = 2R_1,$$

and hence, we have for $\alpha = 1$ (or for $\alpha = 2$)

$$\left| \sum_{i=1}^{N_1} s^{(\alpha)}_i \right| \geq R_1. \quad (21)$$

Let us put $s_1 = s^{(\alpha)}_i$ for $1 \leq i \leq N_1$ and that $\alpha$ for which (21) is valid. We have constructed $s_1$ ($1 \leq i \leq N_1$) as in both case a) and case b) so that (19) is valid and $|A_1| > \sqrt{p_1}$. The first step in the induction is thereby terminated.

**Second Step in the Induction.** For integer $n \geq 2$ let us have constructed $p_n$, $N_{n-1}$, and $s_i$ for $1 \leq i \leq N_{n-1}$. Let us select $p_n$ such that

$$\sum_{x=p_{n-1}+1}^{p_n} \frac{1}{\sqrt{2k}} > 2\sqrt{p_n},$$

and let us find the natural number $N_n$ such that $K(\tau', N_0) > 2p_n$, and the partial sum

$$s_{N_n}(x) = \sum_{i=1}^{N_n} q_{s_i}(x) \quad (22)$$

contains all functions $\{\varphi_1(x)\}_{i=1}^{2p_n}$ and $\{\varphi_{m_1}(x)\}_{i=1}^{N_{n-1}}$ which have been reduced in (22). We will construct the $s_i$ with $N_{n-1} < i \leq N_n$ almost exactly as in the first step of the induction, with the sole exception that if the number $s_i$ ($1 \leq i \leq N_{n-1}$) has been compared to the function $f_{m_1}(x)$ in the partial sum $s_{N_{n-1}}(x)$, then we compare the number $-s_i$ to the function $-f_{m_1}(x)$ in the second step.

The sequence $\{s_i\}_{i=1}^\infty$ thus constructed satisfies the following conditions:

1°) the sum is $\left| \sum_{i=1}^{N_n} s_i \right| > \sqrt{p_n}$;

2°) if the number $s_1 = (1/\sqrt{7})$ has been compared to the function $\varphi_{m_1}(x) = \varphi_{m_2}(x)$ ($1 \leq i < \infty$), then $i < 27 n^2$;

3°) if the number $s_i$ has been compared to the function $f_{m_1}(x)$, then the number $s_j = -s_i$ is compared to the function $-f_{m_1}(x)$;

4°) the sequence $\{s_i\}_{i=1}^\infty$ is some transposition of the sequence $\{\frac{1}{\sqrt{m}}, -\frac{1}{\sqrt{m}}\}_{m=1}^\infty$.

Let us show that there exists a point $x_l \in B$, for which

$q_{m_1}(x_l) = s_1 \quad (1 \leq i < \infty)$.\]

Indeed, by construction a number $s_{l_k} = (1/\sqrt{n_k})$, i.e.,

$q_{m_1}(x) \Rightarrow \frac{1}{\sqrt{n_k}} \quad (1 \leq k < \infty),$

and

$q_{m_{l_k}}(x) \Rightarrow q_{m_1}(x) \Rightarrow -\frac{1}{\sqrt{n_k}}, \quad (23)$

where the sign $\Rightarrow$ denotes comparison, has been compared to each positive function $\varphi_{m_1}(x) = \varphi_{m_{l_k}}(x)$ from the series (10). Let us take the point

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By virtue of \( 2^\circ \) and \( 4^\circ \) the point \( x_1 \in \mathbb{B} \), because \( n_k \leq 27 k^2 \) and \( \{n_k\}_{k=1}^\infty \) are the transposition of a natural series. But then [see (7) and \( 3^\circ \)]

\[
q_{m_i}(x_i) \equiv q_{2i-1}(x_i) = \frac{1}{\sqrt{n_i}} = s_i
\]

Hence [see \( 1^\circ \), (23), and (24)]

\[
\left| \sum_{j=1}^N q_n(x) \right| = \left| \sum_{j=1}^N s_j \right| > \sqrt{p_n} \quad (n = 1, 2, \ldots),
\]

i.e., [see (22)], \( \lim_{n \to \infty} \left| \sigma_n(x) \right| = \infty \). It hence follows that the series (10) diverges at the point \( x_1 \) and the impossibility of case II is thereby proved. The functions \( \mathbb{B} \subset [0, 1] \) with \( 0 \leq m = \inf x < \sup x = M = 1 \). Let \([m, M] - \mathbb{B} = \bigcup_{i=1}^\infty (a_i, b_i)\), where \((a_i, b_i)\) are adjacent intervals in \( \mathbb{B} \). Let us define the functions \( f_k(x) \in C(0, 1) \) follows:

\[
q_{2j-1}(x) \quad \text{for} \quad x \in \mathbb{B},
\]

\[
q_{2j-1}(m) \quad \text{for} \quad x \in [0, m],
\]

\[
q_{2j-1}(M) \quad \text{for} \quad x \in [M, 1],
\]

\[
f_{2j-1}(x) = \begin{cases} 
q_{2j-1}(a_i) & \text{for} \quad x \in [a_i, a_i + \frac{i-1}{i} (b_i - a_i)] \\
\text{linear in} \quad [a_i + \frac{i-1}{i} (b_i - a_i), b_i] & (1 \leq i < \infty), \\
\end{cases}
\]

and let

\[
f_{2j}(x) = -f_{2j-1}(x) \quad (j = 1, 2, \ldots).
\]

The functions \( f_n(x) \) are those desired. Indeed, the series (4) diverges at each point \( x \in [0, 1] \) since \( f_n(x) = g_n(a) \) for \( n \geq N(x) \) for some point \( a = a(x) \in \mathbb{B} \). On the other hand, evidently the series (3) converges to zero at each point \( x \in [0, 1] \), and the set consists only of one function \( f(x) \equiv 0 \). The theorem is proved.

**LITERATURE CITED**