ON A TWO–DIMENSIONAL ANALOG OF SZEMERÉDI’S
THEOREM IN ABELIAN GROUPS

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ABSTRACT

Let $G$ be a finite Abelian group and $A \subseteq G \times G$ be a set of cardinality at least $|G|^2 / (\log \log |G|)^c$, where $c > 0$ is an absolute constant. We prove that $A$ contains a triple $\{(k, m), (k + d, m), (k, m + d)\}$, where $d \neq 0$. This theorem is a two-dimensional generalization of Szemerédi’s theorem on arithmetic progressions.

1. Introduction.

Szemerédi’s theorem [29] on arithmetic progressions states that an arbitrary set $A \subseteq \mathbb{Z}$ of positive density contains arithmetic progression of any length. This remarkable theorem has played a significant role in the development of two fields in mathematics: additive combinatorics (see e.g. [31]) and combinatorial ergodic theory (see e.g. [10]). A more precise statement of the theorem is as follows.

Let $N$ be a natural number. We set

$$a_k(N) = \frac{1}{N} \max\{|A| : A \subseteq \{1, 2, \ldots, N\},$$

$$A \text{ contains no arithmetic progressions of length } k\},$$

where $|A|$ denotes the cardinality of $A$.

**Theorem 1.1** (Szemerédi, 1975). For any $k \geq 3$ the following holds

$$a_k(N) \rightarrow 0 \text{ as } N \rightarrow \infty.$$  \hspace{1cm} (1.1)

Clearly, this result implies van der Waerden’s theorem [33].

In the simplest case $k = 3$ of Theorem 1.1 was proven by K.F. Roth [22] in 1953, who applied the Hardy – Littlewood method to show that

$$a_3(N) \ll \frac{1}{\log \log N}.$$  \hspace{1cm} (1.2)

At present, the best upper bound for $a_3(N)$ is due to J. Bourgain [4]. He proved that

$$a_3(N) \ll \frac{(\log \log N)^2}{(\log N)^{2/3}}.$$  \hspace{1cm} (1.2)

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Szemerédi’s proof uses difficult combinatorial arguments. An alternative proof was suggested by Furstenberg in [10] (see also [10]). His approach uses the methods of ergodic theory. Furstenberg showed that Szemerédi’s theorem is equivalent to the multiple recurrence of almost all points in any dynamical system.

A. Behrend [2] obtained the following lower bound for \( a_3(N) \)

\[
a_3(N) \gg \exp(-C(\log N)^{3/2}),
\]

where \( C \) is an absolute constant. A lower bound on \( a_k(N) \) for an arbitrary \( k \) was given in [21].

Unfortunately, Szemerédi’s methods give very weak upper estimates for \( a_k(N) \). The ergodic approach gives no estimates at all. Only in 2001 W. T. Gowers [11] obtained a quantitative result concerning the rate at which \( a_k(N) \) approaches zero for \( k \geq 4 \). He proved the following theorem.

**Theorem 1.2.** For any \( k \geq 4 \), we have

\[
a_k(N) \ll 1/(\log \log N)^{c_k},
\]

where the constant \( c_k \) depends on \( k \) only.

In paper [1] and book [10] the following problem was considered. Let \( \{1, 2, \ldots, N\}^2 \) be the two–dimensional lattice with basis \( \{(1, 0), (0, 1)\} \). Let also

\[
L(N) = \frac{1}{N^2} \max \{ |A| : A \subseteq \{1, 2, \ldots, N\}^2 \text{ and } A \text{ contains no triples of the form } \{(k, m), (k + d, m), (k, m + d)\} \}
\]

A triple from (1.3) is called a "corner". In [1][10] it was proven that \( L(N) \) tends to 0 as \( N \) tends to infinity. W. T. Gowers (see [11]) asked the question of what is the rate of convergence of \( L(N) \) to 0.

The following theorem was proven in [26][27] (see also [28][32][24][25]).

**Theorem 1.3.** Let \( \delta > 0 \), and \( N \gg \exp(\exp(\delta^{-73})) \). Let also \( A \) be a subset of \( \{1, \ldots, N\}^2 \) of cardinality at least \( \delta N^2 \). Then \( A \) contains a corner.

Thus, we have the estimate \( L(N) \ll 1/(\log \log N)^{1/73} \).

The question on upper estimates for \( L(N) \) in the group \( \mathbb{F}_3^n \) was considered in [15] and [18].

A natural generalization of Theorem 1.3 above is replace \( \{1, \ldots, N\} \) or \( \mathbb{Z}/N\mathbb{Z} \) to an arbitrary Abelian group. Such generalizations of Roth’s theorem and Theorems 1.1, 1.2 were obtained in papers [5][8][20][19][17].

The main result of this paper is the following theorem.

Let \( G \) be a finite Abelian group with additive group operation \(+\). In the case any triple of the form \( \{(k, m), (k + d, m), (k, m + d)\} \), where \( d \neq 0 \) is called a corner.

**Theorem 1.4.** Let \( G \) be a finite Abelian group and \( A \subseteq G \times G \) be a set of cardinality at least \( |G|^2/(\log \log |G|)^c \), where \( c > 0 \) is an absolute constant. Then \( A \) contains a corner.
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Note. The constant $c$ in Theorem 1.4 might be taken as $1/22$.

The proof of Theorem 1.4 is contained in §3,4,5,6 and proceeds by an iteration scheme as in all known effective proofs of Szemerédi–type theorems.

Let $G$ be an Abelian group, $A \subseteq G \times G$, $|A| \gg |G|^2/(\log \log |G|)^c$ and we want to find a corner in $A$. At each step of our procedure we prove the following: either $A$ is "sufficiently regular" or its "density" can be increased. A suitable definition of "sufficiently regular" sets (so–called uniform sets) is one of the main aims of our proof.

If $A$ is a random set and $A$ has cardinality $\delta|G|^2$ then it is easy to see that $A$ contains approximately $\delta^2 N^3$ corners. We shall say $A$ is regular (or in other words $\alpha$–uniform) if $A$ contains the same approximate number of corners.

Let $E_1, E_2$ be subsets of $A$, where $A \subseteq G$ to be chosen later. Let $A$ be a subset of $E_1 \times E_2$ of cardinality $\delta |E_1||E_2|$ . We shall say that $A$ is rectilinearly $\alpha$–uniform if, roughly speaking, the number of quadruples $\{(x, y), (x+d, y), (x, y+s), (x+d, y+s)\}$ in $A^4$ is at most $(\delta^2 + \alpha)|E_1|^2|E_2|$, $\alpha > 0$ (in fact we need a slightly different definition of $\alpha$–uniformity, which depends on the set $A$). In §3 we prove that if $E_1, E_2$ has small Fourier coefficients and $A$ is rectilinearly $\alpha$–uniform then $A$ has about the expected number of corners. Simple observation shows (see e.g. [27]) that the notion of rectilinearly $\alpha$–uniformity cannot be expressed in terms of Fourier transform, more precisely, there is a set, say $A_0$, with really small Fourier coefficients but large number of quadruples $\{(x, y), (x+d, y), (x, y+s), (x+d, y+s)\} \in A_0^4$. On the other hand, we can define a rectilinearly $\alpha$–uniform set using so–called rectilinear norm (see §3).

Suppose that $A$ fails to be rectilinearly $\alpha$–uniform. Roughly speaking, it means that $A$ has no random properties. The last observation can be expressed precisely by showing that $A$ has increased density $\delta + c(\delta)$, $c(\delta) > 0$ on some product set $F_1 \times F_2$, $F_1 \subseteq E_1$, $F_2 \subseteq E_2$ (see §4). Clearly, this density increment can only occur finitely many times, because the density of any set does not exceed one. Thus, our iteration scheme must stop after finite number of steps. It means that we find a rectilinear $\alpha$–uniform subset of the set $A$ and consequently a corner in $A$.

Unfortunately, the structure of $F_1 \times F_2$ need not be regular and we cannot make the next step of our procedure directly. To make $F_1 \times F_2$ regular, we pass to a subset of $A$, say, $A'$ and a vector $t = (t_1, t_2) \in G \times G$ such that $(F_1 - t_1) \cap A'$, $(F_2 - t_2) \cap A'$ has small Fourier coefficients.

We are now in the situation we started with, but $A$ has a larger density and we iterate the procedure. This also can only occur finitely many times. In §6 we combine the arguments from the earlier sections and show that they give the bound that we stated in Theorem 1.4.

In our prove we chose $A$ to be a Bohr set (see [3] [4] [14] and others). Note that the best upper bound for $a_3(N)$ was proven by J. Bourgain in [4] using exactly these very sets. The properties of Bohr sets will be considered in §2.

The constructions which we use develop the approach of [3] [11] [24] [27]. We improve our constant $c$ by more accurate calculations than in [27].

In our forthcoming papers we are going to obtain a multidimensional analog of Theorem 1.4.

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2. **On Bohr sets.**

Let $G = (G,+)$ be a finite Abelian group with additive group operation $+$. Suppose that $A$ is a subset of $G$. It is very convenient to write $A(x)$ for such a function. Thus $A(x) = 1$ if $x \in A$ and $A(x) = 0$ otherwise. By $\hat{G}$ denote the Pontryagin dual of $G$, in other words the space of homomorphisms $\xi$ from $G$ to $\mathbb{T}$, $\xi : x \mapsto \xi \cdot x$. It is well known that $\hat{G}$ is an additive group which is isomorphic to $G$. Also denote by $N$ the cardinality of $G$.

One of the crucial moments in [3] was the notion of Bohr set.

Let $S$ be a subset of $\hat{G}$, $|S| = d$, $\varepsilon > 0$ be a real number.

**Definition 2.1.** Define the Bohr set $\Lambda = \Lambda(S, \varepsilon)$ by

$$\Lambda(S, \varepsilon) = \{ n \in G \mid \| \xi \cdot n \| < \varepsilon \text{ for all } \xi \in S \}.$$  

We shall say that the set $S \subseteq \hat{G}$ is *generative set* of Bohr set $\Lambda$. The number $d$ is called *dimension* of Bohr set $\Lambda$ and is denoted by $\dim \Lambda$. If $M = \Lambda + n$, $n \in G$ is a translation of $\Lambda$, then, by definition, put $\dim M = \dim \Lambda$.

Another construction of Bohr set (so-called *smoothed* Bohr set) was given in [30] and [14].

**Definition 2.2.** Let $0 < \kappa < 1$ be a real number. A Bohr set $\Lambda = \Lambda(S, \varepsilon)$ is called *regular*, if for an arbitrary $\varepsilon'$ such that

$$|\varepsilon - \varepsilon'| < \frac{\kappa}{100d} \varepsilon$$

we have

$$1 - \kappa < \frac{|\Lambda(S, \varepsilon')|}{|\Lambda(S, \varepsilon)|} < 1 + \kappa.$$

We need several results concerning Bohr sets (see [3] and [14]).

**Lemma 2.1.** Let $\Lambda(S, \varepsilon)$ be a Bohr set, $|S| = d$. Then

$$|\Lambda(S, \varepsilon)| \geq \varepsilon^d N.$$  

**Lemma 2.2.** Let $0 < \kappa < 1$ be a real number, and $\Lambda(S, \varepsilon)$ be a Bohr set. Then there exists $\varepsilon_1$ such that $\frac{\kappa}{2} < \varepsilon_1 < \varepsilon$ and such that $\Lambda(S, \varepsilon_1)$ is a regular Bohr set.

*All Bohr sets will be regular in the article.*

**Definition 2.3.** Let $f, g$ be functions from $G$ to $\mathbb{C}$. By $f * g$ define the function

$$(f * g)(n) = \sum_{s \in G} f(s)g(n - s).$$

**Definition 2.4.** Let $\varepsilon \in (0,1]$ be a real number, and $\Lambda(S, \varepsilon_0)$ be a Bohr set, $S \subseteq \hat{G}$, $|S| = d$. A regular Bohr set $\Lambda' = \Lambda(S', \varepsilon')$ is called an $\varepsilon$-attendant of $\Lambda$ if $S \subseteq S'$ and $\varepsilon \varepsilon_0 / 2 \leq \varepsilon' \leq \varepsilon \varepsilon_0$. 
Lemma 2.2 implies that for an arbitrary Bohr set there exists its \( \varepsilon \)-attendant. We shall consider that \( S' = S \) unless stated otherwise.

Let \( n \) be an arbitrary element of the group \( G \), and \( \Lambda \) be a Bohr set. We shall say that a Bohr set \( \Lambda' \) is an \( \varepsilon \)-attendant of \( \Lambda + n \), if \( \Lambda' \) is an \( \varepsilon \)-attendant of \( \Lambda \).

The following lemma is also due to J. Bourgain. We give his proof for the sake of completeness.

**Lemma 2.3.** Let \( \kappa > 0 \) be a real number, \( S \subseteq \hat{G} \), \( \Lambda = \Lambda(S, \varepsilon) \) be a regular Bohr set, and \( \Lambda' = \Lambda(S, \varepsilon') \) its \( \kappa/(100d) \)-attendant. Then the number of \( n \)'s such that \( (\Lambda \ast \Lambda')(n) > 0 \) does not exceed \( |\Lambda| (1 + \kappa) \), the number of \( n \)'s such that \( (\Lambda \ast \Lambda')(n) = |\Lambda'| \) is greater than \( |\Lambda| (1 - \kappa) \) and

\[
\left\| \frac{1}{|\Lambda'|}(\Lambda \ast \Lambda')(n) - \Lambda(n) \right\|_1 < 2\kappa|\Lambda|.
\]

**(2.1)**

**Proof.** If \( (\Lambda \ast \Lambda')(n) > 0 \) then there exists \( m \) such that for any \( \xi \in S \), we have

\[
\|\xi \cdot m\| < \frac{\kappa}{100d}\varepsilon, \quad \|\xi \cdot (n - m)\| < \varepsilon.
\]

Using (2.2), we get for all \( \xi \in S \)

\[
\|\xi \cdot n\| < \left(1 + \frac{\kappa}{100d}\right)\varepsilon,
\]

for all \( \xi \in S \). It follows that

\[
n \in \Lambda^+ := \Lambda \left( S, \left(1 + \frac{\kappa}{100d}\right)\varepsilon \right).
\]

By Lemma 2.2 we have \( |\Lambda^+| \leq (1 + \kappa)|\Lambda| \).

On the other hand, if

\[
n \in \Lambda^- := \Lambda \left( S, \left(1 - \frac{\kappa}{100d}\right)\varepsilon \right)
\]

then \( (\Lambda \ast \Lambda')(n) = |\Lambda'| \). Using Lemma 2.2 we obtain \( |\Lambda^-| \geq (1 - \kappa)|\Lambda| \).

Let us prove (2.1). We have

\[
\left\| \frac{1}{|\Lambda'|}(\Lambda \ast \Lambda')(n) - \Lambda(n) \right\|_1 = \left\| \frac{1}{|\Lambda'|}(\Lambda \ast \Lambda')(n) - \Lambda(n) \right\|_{L^1(\Lambda^+ \setminus \Lambda^-)}
\]

\[
\leq |\Lambda^+| - |\Lambda^-| < 2\kappa|\Lambda|
\]

as required.

**Corollary.** Lemma 2.3 implies that \( |\Lambda| \leq |\Lambda + \Lambda'| \leq (1 + 2\kappa)|\Lambda| \).

**Note.** Let \( \Lambda^x(n) = \Lambda(n - x) \). Since \( (\Lambda^x \ast \Lambda')(n) = (\Lambda \ast \Lambda')(n - x) \), it follows that (2.1) takes place for translations \( \Lambda + x \).

**Definition 2.5.** By \( \Lambda^+ \) and \( \Lambda^- \) denote the Bohr sets defined in (2.4) and (2.5), respectively, \( \Lambda^- \subseteq \Lambda \subseteq \Lambda^+ \).

By Lemma 2.3 we have \( |\Lambda^+| \leq |\Lambda|(1 + \kappa) \) and \( |\Lambda^-| \geq |\Lambda|(1 - \kappa) \). Note that for any \( s \in \Lambda' \), we get \( \Lambda^- \subseteq \Lambda + s \).

Suppose \( \Lambda \subseteq G \) is a Bohr set, and \( \vec{x} = (x_1, x_2) \) belongs to \( G \times G \). By \( \Lambda + \vec{x} \)
denote the set \((\Lambda + x_1) \times (\Lambda + x_2) \subseteq G \times G\). Let \(\vec{n} \in G \times G\). Let \(\Lambda(\vec{n})\) denote the characteristic function of \(\Lambda \times \Lambda\). We shall write \(\vec{s} \in \Lambda\), \(\vec{s} = (s_1, s_2)\), if \(s_1 \in \Lambda\) and \(s_2 \in \Lambda\).

**Lemma 2.4.** Suppose \(\Lambda\) is a Bohr set, \(\Lambda'\) is its \(\varepsilon\)-attendant, \(\varepsilon = \kappa/(100d)\), \(\vec{x}\) is a vector, and \(E \subseteq G \times G\). Then
\[
\left| \delta_{\Lambda + \vec{x}}(E) - \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda + \vec{x}} \delta_{\Lambda' + \vec{n}}(E) \right| \leq 4\kappa.
\]
(2.6)

**Proof.** We have
\[
\sigma = \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda + \vec{x}} \delta_{\Lambda' + \vec{n}}(E) = \frac{1}{|\Lambda|^2|\Lambda'|^2} \sum_{\vec{s}} E(\vec{s}) \sum_{\vec{n}} \Lambda(\vec{n} - \vec{x}) \Lambda'(\vec{s} - \vec{n})
\]
\[
= \frac{1}{|\Lambda|^2|\Lambda'|^2} \sum_{\vec{s}} E(\vec{s}) \sum_{\vec{n}} \Lambda(\vec{n}) \Lambda'(\vec{s} - \vec{x} - \vec{n})
\]
Using Lemma 2.3 we get
\[
\sigma = \frac{1}{|\Lambda|^2} \sum_{\vec{s}} E(\vec{s}) \Lambda(\vec{s} - \vec{x}) + 4\vartheta \kappa = \delta_{\Lambda + \vec{x}}(E) + 4\vartheta \kappa,
\]
where \(|\vartheta| \leq 1\). This completes the proof.

**Note.** Clearly, the one–dimension analog of Lemma 2.4 takes place.

Let \(\Lambda_1 = \Lambda(S_1, \varepsilon_1)\), \(\Lambda_2 = \Lambda(S_2, \varepsilon_2)\) be two Bohr sets, \(S_1, S_2 \subseteq \hat{G}\). We shall write \(\Lambda_1 \leq \Lambda_2\), if \(S_1 \subseteq S_2\) and \(\varepsilon_1 \leq \varepsilon_2\).

### 3. On \(\alpha\)-uniformity.

Let \(f\) be a function from \(G\) to \(C\), \(N = |G|\). By \(\hat{f}(\xi)\) denote the Fourier transformation of \(f\)
\[
\hat{f}(\xi) = \sum_{x \in G} f(x)e(-\xi \cdot x),
\]
where \(e(x) = e^{2\pi ix}\). We shall use the following basic facts
\[
\sum_{x \in G} |f(x)|^2 = \frac{1}{N} \sum_{\xi \in \hat{G}} |\hat{f}(\xi)|^2.
\]
(3.1)
\[
\sum_{x \in G} f(x)\overline{g(x)} = \frac{1}{N} \sum_{\xi \in \hat{G}} \hat{f}(\xi)\overline{\hat{g}(\xi)}.
\]
(3.2)
\[
\sum_{y \in G} \sum_{x \in G} |f(x)g(y - x)|^2 = \frac{1}{N} \sum_{\xi \in \hat{G}} |\hat{f}(\xi)|^2 |\hat{g}(\xi)|^2.
\]
(3.3)

Let \(\Lambda\) be a Bohr set, and \(A\) be an arbitrary subset of \(\Lambda\). Let \(|A| = \delta|\Lambda|\). Define the balanced function of \(A\) to be \(f(s) = (A(s) - \delta)\Lambda(s) = A(s) - \delta \Lambda(s)\).

Let \(D\) denote the closed disk of radius 1 centered at 0 in the complex plane. Let \(R\)
be an arbitrary set. We write $f : R \to D$ if $f$ is zero outside $R$.
The following definition is due to Gowers [11].

**Definition 3.1.** A function $f : \Lambda \to D$ is called $\alpha$–uniform if
$$\|\hat{f}\|_\infty \leq \alpha|\Lambda|.$$
(3.4)

We say that $A$ is $\alpha$–uniform if its balanced function is.

Let us prove an analog of Lemma 2.2 from [11].

**Lemma 3.1.** Let $\Lambda$ be a Bohr set, and let $f : \Lambda \to D$ be an $\alpha$–uniform function.
Then we have
$$\sum_k |\sum_s f(s)g(k-s)|^2 \leq \alpha^2|\Lambda|^2\|g\|_2^2,$$
for an arbitrary function $g$, $g : G \to D$.

**Proof.** By (3.3) we get
$$\sum_k |\sum_s f(s)g(k-s)|^2 = \sum_\xi |\hat{f}(\xi)|^2|\hat{g}(\xi)|^2.$$
(3.5)

Since the function $f$ is $\alpha$–uniform, it follows that $\|\hat{f}\|_\infty \leq \alpha|\Lambda|$. Using this inequality and (3.2), we have
$$\sum_k |\sum_s f(s)g(k-s)|^2 \leq \alpha^2|\Lambda|^2\frac{1}{N} \sum_\xi |\hat{g}(\xi)|^2 = \alpha^2|\Lambda|^2\|g\|_2^2.$$
(3.6)

This completes the proof.

**Corollary 3.1.** Let $S \subseteq G$ be a set, and $\Lambda'$ be a Bohr set. Suppose $E \subseteq \Lambda'$ is $\alpha$–uniform, and $E$ have the cardinality $\delta|\Lambda'|$. Let $g$ be a function from $S$ to $[-1,1]$. Then for all but $\alpha^{2/3}|S|$ choices of $k$ we have
$$|(E * g)(k) - \delta(\Lambda' * g)(k)| \leq \alpha^{2/3}|\Lambda'|.$$

**Proof.** Let $f$ be the balanced function of $E \cap \Lambda'$. Using Lemma 3.1, we get
$$\sum_k |(E * g)(k) - \delta(\Lambda' * g)(k)|^2 = \sum_k |\sum_s f(s)g(k-s)|^2 \leq \alpha^2|\Lambda'|^2\|g\|_2^2 \leq \alpha^2|\Lambda'|^2 |S|.$$
(3.7)

This concludes the proof.

Let $\Lambda_1$ and $\Lambda_2$ be Bohr sets, and $E_1 \times E_2$ be a subset of $\Lambda_1 \times \Lambda_2$. Suppose $f : \Lambda_1 \times \Lambda_2 \to D$ is a function.

**Definition 3.2.** Let $\alpha$ be a real number, $\alpha \in [0,1]$. A function $f : E_1 \times E_2 \to D$
is called rectilinearly $\alpha$–uniform if
\[ \sum_{x,x',y,y'} f(x,y)f(x',y')f(x',y)f(x,y') \leq \alpha |E_1|^2 |E_2|^2. \] (3.9)

Note that the function $f$ is $\alpha$–uniform iff
\[ \sum_{m,p} \left| \sum_k f(k,m)f(k,p) \right|^2 \leq \alpha |E_1|^2 |E_2|^2. \] (3.10)

Let $A$ be a subset of $E_1 \times E_2$, $|A| = \delta |E_1||E_2|$. Define the balanced function of $A$ to be $f(x,y) = (A(x,y) - \delta) \cdot (E_1 \times E_2)(x,y)$. We say that $A \subseteq E_1 \times E_2$ is rectilinearly $\alpha$–uniform if its balanced function is.

Let $f$ be an arbitrary function, $f : G \times G \to C$. Define $\|f\|$ by the formula
\[ \|f\| = \left| \sum_{x,x',y,y'} f(x,y)f(x',y)f(x',y') \right|^{1/4} \] (3.11)

**Lemma 3.2.** $\| \cdot \|$ is a norm.

**Proof.** See [24].

**Definition 3.3.** Let $\Lambda$ be a Bohr set, $Q \subseteq \Lambda$, $|Q| = \delta|\Lambda|$, $\alpha, \varepsilon$ are positive numbers. A set $Q$ is called $(\alpha, \varepsilon)$–uniform if there exists $\Lambda'$ such that $\Lambda'$ is an $\varepsilon$–attendant set of $\Lambda$ and the set
\[ B := \{ m \in \Lambda \mid \| (Q \cap (\Lambda' + m) - \delta(\Lambda' + m)) \|_\infty \geq \alpha |\Lambda'| \} \]
has the cardinality at most $\alpha|\Lambda|$
\[ |B| \leq \alpha|\Lambda|, \] (3.12)
further
\[ \frac{1}{|\Lambda|} \sum_{m \in \Lambda} |\delta_{\Lambda' + m}(Q) - \delta|^2 \leq \alpha^2. \] (3.13)
and
\[ \| (Q \cap \Lambda - \delta \Lambda') \|_\infty \leq \alpha |\Lambda|. \] (3.14)

Certainly, this definition depends on $\Lambda$ and $\Lambda'$. We do not assume that $\Lambda'$ has the same generative set as $\Lambda$. If $Q$ is $(\alpha, \varepsilon)$–uniform and $\Lambda'$ is an $\varepsilon$–attendant set of $\Lambda$ then we shall mean sometimes that $\Lambda'$ is an $\varepsilon$–attendant set of $\Lambda$ such that (3.12) — (3.14) hold.

**Note.** Let
\[ B^* = \{ m \in \Lambda \mid |\delta_{\Lambda' + m}(Q) - \delta| \geq \alpha^{2/3} \}. \]
Condition (3.13) implies that $|B^*| \leq \alpha^{2/3}|\Lambda|$.

**Note.** Condition (3.14) is not so important as (3.12) and (3.13). The inequality
\[ \| (Q \cap \Lambda - \delta \Lambda') \|_\infty \leq 4\alpha |\Lambda| \]
follows from (3.12), (3.13) (see Proposition 3.1).
Let $\Lambda_1, \Lambda_2$ be Bohr sets, $\Lambda_1 \subseteq \Lambda_2$, $\varepsilon > 0$ be a real number. Let also $E_1, E_2$ be subsets of $\Lambda_1, \Lambda_2$, respectively, and $|E_1| = \beta_1|\Lambda_1|$, $E_2 = \beta_2|\Lambda_2|$. 

**Definition 3.4.** A function $f : E_1 \times E_2 \rightarrow D$ is called rectilinearly $(\alpha, \varepsilon)$-uniform if there exists $\Lambda'$ such that $\Lambda'$ is an $\varepsilon$-attendant of $\Lambda_1$ and

$$\|f\|_{\Lambda_1 \times \Lambda_2, \varepsilon}^4 = \sum_{i \in \Lambda_1} \sum_{j \in \Lambda_2} \sum_{k} \sum_{m,u} \Lambda'(m-k-i)\Lambda'(u-k-i) \times$$

$$|\sum_{r} \Lambda'(k+r-j)f(r,m)f(r,u)|^2 \leq \alpha \beta_1^2 \beta_2^2 |\Lambda'|^4 |\Lambda_1|^2 |\Lambda_2|.$$  \hfill (3.15)

Let $\Lambda_1, \Lambda_2$ be Bohr sets, $\Lambda_1 \subseteq \Lambda_2$. Let also $E_1, E_2$ be subsets of $\Lambda_1, \Lambda_2$, respectively, and $|E_1| = \beta_1|\Lambda_1|$, $E_2 = \beta_2|\Lambda_2|$. 

**Definition 3.5.** Let $A \subseteq E_1 \times E_2$, $|A| = \delta \beta_1 \beta_2 |\Lambda_1||\Lambda_2|$, and $f(\vec{s}) = A(\vec{s}) - \delta(E_1 \times E_2)(\vec{s})$. $A$ is called rectilinearly $(\alpha, \alpha_1, \varepsilon)$-uniform if there exist $\Lambda', \Lambda'_c$ such that $\Lambda'$ is an $\varepsilon$-attendant of $\Lambda_1$, $\Lambda'_c$ is an $\varepsilon$-attendant of $\Lambda'$ and the set

$$B = \{l \in \Lambda_1 \mid \|f_l\|_{\Lambda'_c \times \Lambda_2, \varepsilon}^4 > \alpha_1^2 \beta_1^2 \beta_2^2 |\Lambda'_c|^4 |\Lambda'|^2 |\Lambda_2|\},$$

where $f_l(\vec{s}) := f(s_1 + l, s_2)\Lambda'(s_1)$, $l \in \Lambda_1$ has the cardinality at most $\alpha_1|\Lambda_1|$.

Note that

$$\|f_l\|_{\Lambda'_c \times \Lambda_2, \varepsilon}^4 = \sum_{i \in \Lambda'} \sum_{j \in \Lambda_2} \sum_{k} \sum_{m,u} \Lambda''(m-k-i)\Lambda''(u-k-i) \times$$

$$\sum_{r} \Lambda''(k+r-j)f_l(r,m)f_l(r,u)|^2,$$

where $\Lambda'' = \Lambda'_c$ and $\tilde{f}$ is a restriction of $f$ to $(\Lambda' + l) \times \Lambda_2$.

**Note.** We need parameter $\alpha_1$ to decrease the constant $c$ in Theorem 3.4. To obtain Theorem 3.4, with $c$ equals, say, 1000, one can put $\alpha_1 = \alpha$.

**Lemma 3.3.** Let $\Lambda$ be a Bohr set. Suppose $\Lambda'$ is an $\varepsilon$-attendant of $\Lambda$, $\Lambda''$ is an $\varepsilon$-attendant $\Lambda'$ and an $\varepsilon^2$-attendant of $\Lambda$, $\varepsilon = \alpha^2/100d$, $Q \subseteq \Lambda$, $|Q| = \delta\Lambda$, and $\alpha > 0$. Let

$$\Omega_1 = \{s \in \Lambda \mid |\delta_{\Lambda'+s}(Q) - \delta| \geq 4\alpha^{1/2} \text{ or } \frac{1}{|\Lambda|} \sum_{n \in \Lambda'} |\delta_{\Lambda''+n}(Q) - \delta|^2 \geq 4\alpha^{1/2}\}.$$ 

$$\Omega_2 = \{s \in \Lambda \mid ||(Q \cap (\Lambda' + s) - \delta(\Lambda' + s))\|_\infty \geq 4\alpha^{1/2}|\Lambda'|\}.$$ 

1) If

$$\frac{1}{|\Lambda|} \sum_{n \in \Lambda} |\delta_{\Lambda''+n}(Q) - \delta|^2 \leq \alpha^2,$$ \hfill (3.16)

then $|\Omega_1| \leq 4\alpha^{1/2}|\Lambda|$. 

2) If

$$\Omega' = \{s \in \Lambda \mid ||(Q \cap (\Lambda'' + s) - \delta(\Lambda'' + s))\|_\infty \geq \alpha|\Lambda''|\}$$ \hfill (3.17)

has the cardinality at most $\alpha|\Lambda|$, then $|\Omega_2| \leq 4\alpha^{1/2}|\Lambda|$. 


3) Suppose $Q$ is $(\alpha, \varepsilon^2)$–uniform subset of $\Lambda$ and $\Lambda''$ is an $\varepsilon^2$–attendant of $\Lambda$ such that \((3.12) \rightarrow (3.14)\) hold. Let

$$\tilde{\Omega} = \{s \in \Lambda \mid (Q - s) \cap \Lambda' \text{ is not } (8\alpha^{1/4}, \varepsilon)\text{–uniform}\}.$$ 

Then $|\tilde{\Omega}| \leq 8\alpha^{1/2} |\Lambda|$. 

**Proof.** Let us prove 1). Let $\delta'_n = \delta_{\Lambda' + n}(Q)$, $\delta''_n = \delta_{\Lambda'' + n}(Q)$, $\kappa = \alpha^2/4$, and $\varepsilon = \alpha^{1/2}$. Consider the sets

$$B_s = \{n \in \Lambda' + s \mid |\delta''_n - \delta| \geq \varepsilon\}, \quad G_s = \{n \in \Lambda' + s \mid |\delta''_n - \delta| < \varepsilon\}, \quad s \in \Lambda$$

and the sets

$$B = \{s \in \Lambda \mid |B_s| \geq \varepsilon|\Lambda'|\}, \quad G = \{s \in \Lambda \mid |B_s| < \varepsilon|\Lambda'|\}.$$ 

If $s \in G$ then $|B_s| < \varepsilon|\Lambda'|$. Using Lemma 2.4 we have

$$|\delta'_n - \delta| \leq \frac{1}{|\Lambda|} \sum_{x \in \Lambda' + s} |\delta''_n - \delta| + 4\kappa \leq \frac{1}{|\Lambda|} \sum_{x \in \Lambda' + s} |\delta''_n - \delta| + 4\kappa \leq \frac{1}{|\Lambda'|} \sum_{x \in B_s} |\delta''_n - \delta| + \frac{1}{|\Lambda'|} \sum_{x \in G_s} |\delta''_n - \delta| + 4\kappa < \varepsilon + \varepsilon = 2\varepsilon. \quad (3.18)$$

Besides that for $s \in G$, we get

$$\frac{1}{|\Lambda'|} \sum_{x \in \Lambda' + s} |\delta''_n - \delta|^2 \leq \frac{1}{|\Lambda'|} \sum_{x \in B_s} |\delta''_n - \delta|^2 + \frac{1}{|\Lambda'|} \sum_{x \in G_s} |\delta''_n - \delta|^2 \leq \varepsilon^2 + \varepsilon^2 \leq 2\varepsilon. \quad (3.19)$$

Let us estimate the cardinality of $B$. We have

$$\alpha^2 \geq \frac{1}{|\Lambda|} \sum_{s \in B} |\delta''_n - \delta|^2 \geq \frac{1}{|\Lambda| |\Lambda'|} \sum_{s \in B} \sum_{n \in \Lambda' + s} |\delta''_n - \delta|^2 - 4\kappa \geq \frac{1}{|\Lambda| |\Lambda'|} \sum_{n \in B_s} |\delta''_n - \delta|^2 - 4\kappa \geq \frac{|B| \varepsilon|\Lambda'|}{|\Lambda'| |\Lambda'|} - 4\kappa. \quad (3.18)$$

It follows that $|B| \leq 4\alpha^{1/2} |\Lambda|$. Using (3.18), (3.19) we get $\Omega_1 \subseteq B$ and 1) is proven.

To prove 2) it suffices to note that

$$\frac{1}{|\Lambda| |\Lambda'|} \sum_{s \in \Lambda} \|Q \cap (\Lambda'' + s) - \delta(\Lambda'' + s)\|_{\infty} = \frac{1}{|\Lambda| |\Lambda'|} \sum_{s \in \Omega^*} \|Q \cap (\Lambda'' + s) - \delta(\Lambda'' + s)\|_{\infty} + \frac{1}{|\Lambda| |\Lambda'|} \sum_{s \in (\Lambda \setminus \Omega^*)} \|Q \cap (\Lambda'' + s) - \delta(\Lambda'' + s)\|_{\infty} \leq \alpha + \frac{\alpha|\Lambda'|}{|\Lambda| |\Lambda'|} |\Lambda \setminus \Omega^*| \leq 2\alpha.$$ 

and define the sets $B'_s, G'_s, B', G'$:

$$B'_s = \{n \in \Lambda' + s \mid \|Q \cap (\Lambda'' + n) - \delta(\Lambda'' + n)\|_{\infty} \geq \epsilon_1|\Lambda''|\}, \quad G'_s = \{n \in \Lambda' + s \mid \|Q \cap (\Lambda'' + n) - \delta(\Lambda'' + n)\|_{\infty} < \epsilon_1|\Lambda''|, \quad s \in \Lambda.$$ 

$$B' = \{s \in \Lambda \mid |B_s| \geq \epsilon_1|\Lambda'|\} \quad \text{and} \quad G' = \{s \in \Lambda \mid |B_s| < \epsilon_1|\Lambda'|\},$$

where $\epsilon_1 = \alpha^{1/4}$. After that we can apply the same arguments as above, using Lemma 2.3 instead of Lemma 2.4.

Let us prove 3). Since $Q$ is $(\alpha, \varepsilon^2)$–uniform subset of $\Lambda$, it follows that $Q$ satisfies
Also we have $|\Omega^*| \leq \alpha|\Lambda|$, and $|B|, |B'| \leq 4\alpha^{1/2}|\Lambda|$ (see above). It is easily shown that for all $s \notin B \cup B'$ the set $(Q - s) \cap \Lambda'$ is $(8\alpha^{1/4}, \varepsilon)$–uniform. This completes the proof.

In the same way we can prove

**Proposition 3.1.** Let $\Lambda$ be a Bohr set, and $E \subseteq \Lambda$, $|Q| = \delta|\Lambda|$ be $(\alpha, \varepsilon)$–uniform, $\varepsilon = \alpha/4(100d)$. Then

\[\|(Q \cap \Lambda - \delta\Lambda')\|_\infty < 4\alpha|\Lambda|.
\]

(3.20)

We will not, however, use this fact.

Let $\Lambda_1, \Lambda_2$ be Bohr sets, $\Lambda_1 \leq \Lambda_2$, and $E_1 \subseteq \Lambda_1, E_2 \subseteq \Lambda_2$, $|E_1| = \beta_1|\Lambda_1|, |E_2| = \beta_2|\Lambda_2|$. By $\mathcal{P}$ denote the $E_1 \times E_2$. Let $A \subseteq \mathcal{P}$, $|A| = \delta|E_1||E_2|$. Denote by $H$ and $W$ two copies of the set $A$.

**Theorem 3.1.** Let $f : \mathcal{P} \to \mathbb{D}$ be a rectilinearly $(\alpha, \varepsilon)$–uniform function. Suppose that sets $E_1, E_2$ are $(\alpha_0, \varepsilon)$–uniform, $\alpha_0 = 2^{-50}\alpha^2\beta_2^{1/2}$, $\varepsilon = 2^{-10}\varepsilon_0$, $\varepsilon_0 = (2^{-10}\alpha_0^2)/(100d)$. Let $\Lambda_1$ be an $\varepsilon_0$–attendant of $\Lambda_2$. Then either

\[|\sum_{s_1, s_2, r} H(s_1, s_2)W(s_1 + r, s_2 + r)f(s_1, s_2 + r)| \leq 16\alpha^{1/4}\delta^{3/4}\beta_2^2|\Lambda_1|^2|\Lambda_2|
\]

(3.21)

or there exists a Bohr set $\Lambda'$, two sets $F_1, F_2$ and a vector $\vec{y} = (y_1, y_2) \in G \times G, F_1 \subseteq E_1 \cap (\Lambda' + y_1), F_2 \subseteq E_2 \cap (\Lambda' + y_2)$ such that $\Lambda'$ is an $\varepsilon_0$–attendant of $\Lambda_1$ and

\[|F_1| \geq 2^{-2}\beta_1|\Lambda'|, |F_2| \geq 2^{-2}\beta_2|\Lambda'| \quad \text{and}
\]

\[\delta_{F_1 \times F_2}(A) \geq 2\delta.
\]

(3.22)

(3.23)

**Proof.** Let $\Lambda'$ be an $\varepsilon_0$–attendant of $\Lambda_1$ to be chosen later. Let

\[\Omega_1^{(1)} = \{s \in \Lambda_1 \mid \|(E_1 \cap (\Lambda' + s) - \delta(\Lambda' + s))\|_\infty \geq \alpha_0\},\]

\[\Omega_2^{(1)} = \{s \in \Lambda_1 \mid \|\delta_{\Lambda + s}(E_1) - \beta_1\| \geq \alpha_0^{2/3}\},\]

and

\[\Omega_1^{(2)} = \{s \in \Lambda_2 \mid \|(E_2 \cap (\Lambda' + s) - \delta(\Lambda' + s))\|_\infty \geq \alpha_0\},\]

\[\Omega_2^{(2)} = \{s \in \Lambda_2 \mid \|\delta_{\Lambda + s}(E_2) - \beta_2\| \geq \alpha_0^{2/3}\},\]

Let also $\Omega_1 = \Omega_1^{(1)} \cup \Omega_1^{(2)}$, and $\Omega_2 = \Omega_2^{(1)} \cup \Omega_2^{(2)}$. By assumption the sets $E_1, E_2$ are $(\alpha_0, \varepsilon)$–uniform. Let $\Lambda'$ be $\varepsilon_0$–attendant of $\Lambda_1$ such that (3.12) – (3.14) hold. Using definitions and Lemma 3.3 we get $|\Omega_1^{(1)}| \leq \alpha_0^{2/3}|\Lambda_1|$, $|\Omega_2^{(2)}| \leq \alpha_0^{2/3}|\Lambda_2|$, $l = 1, 2$. Hence $|\Omega_1| \leq 2\alpha_0^{2/3}|\Lambda_1|$ and $|\Omega_2| \leq 2\alpha_0^{2/3}|\Lambda_2|$.

Let $g_i(s) = g_i(k, m) = W(k, m)\Lambda'(k - i), i \in \Lambda_1$, and $h_j(s) = h_j(k, m) = H(k, m)\Lambda'(m - j), j \in \Lambda_2$. We have $k \in \Lambda_1, m \in \Lambda_2$ and $k + r \in \Lambda_1$ in (3.21). It follows that the sum (3.24) does not exceed $|\Lambda_1|^2|\Lambda_2|$. Let also $\lambda_i = \Lambda' + i$, and
\( \mu_j = \Lambda' + j \). Using Lemma 2.3 we get
\[
\sigma_0 = \sum_{s_1, s_2, r} H(s_1, s_2)W(s_1 + r, s_2 + r)f(s_1, s_2 + r) = \\
\sum_{k, m} \sum_r H(k, m)W(k + r, m + r)f(k, m + r)\Lambda_1(k + r)\Lambda_2(m) = \\
\frac{1}{|\Lambda'|^2} \sum_{k, m} \sum_r H(k, m)W(k + r, m + r)f(k + r, m + r)(\Lambda_1*\Lambda')(k + r)(\Lambda_2*\Lambda')(m) + 16\vartheta_0\kappa|\Lambda_1|^2|\Lambda_2| = \\
\frac{1}{|\Lambda'|^2} \sum_{i\in\Lambda_1} \sum_{j\in\Lambda_2} \sum_{k, m} \sum_r h_j(k, m)g_i(k, m + r)f(k, m + r) + 16\vartheta_0\kappa|\Lambda_1|^2|\Lambda_2|, \quad (3.24)
\]
where \( |\vartheta_0| \leq 1 \) and \( \kappa \leq 2^{10}\alpha_0^2 \). Split the sum \( \sigma_0 \) as
\[
\sigma_0 = \tilde{\sigma}_0 + \sigma_0' + \sigma_0'' + \sigma_0'''+ R, \quad (3.25)
\]
The sum \( \tilde{\sigma}_0 \) is taken over \( i \notin \Omega_1, j \notin \Omega_2 \), the sum \( \sigma_0' \) is taken over \( i \in \Omega_1, j \notin \Omega_2 \), the sum \( \sigma_0'' \) is taken over \( i \notin \Omega_1, j \in \Omega_2 \), the sum \( \sigma_0''' \) is taken over \( i \in \Omega_1, j \in \Omega_2 \) and \( |R| \leq 16\varepsilon|\Lambda_1|^2|\Lambda_2| \). Let us estimate \( \sigma_0, \sigma_0' \) and \( \sigma_0'' \). Rewrite \( \sigma_0 \) as
\[
\sigma_0 = \frac{1}{|\Lambda'|^2} \sum_{i\in\Lambda_1} \sum_{j\in\Lambda_2} \sum_{k, m} \sum_r h_j(k - r, m)g_i(k, m + r)f(k - r, m + r) + R. \quad (3.26)
\]
Let \( i \) and \( j \) in the sum (3.26) be fixed. We have \( k \in \lambda_i \) and \( m \in \mu_j \). Further if \( f(k - r, m + r) \) is not zero, then \( k - r \in \Lambda_1 \). It follows that \( r \in \lambda_i - \Lambda_1 = \Lambda' - \Lambda_1 + i \). The set \( \Lambda' \) is \( \varepsilon_0 \)-attendant of \( \Lambda_1 \). Using Lemma 2.3, we obtain that \( r \) belongs to a set of cardinality at most \( 2|\Lambda_1| \). Hence
\[
|\sigma_0''| \leq \frac{1}{|\Lambda'|^2} \sum_{i\in\Lambda_1} |\cdot| \cdot |\Lambda'|^2|\Lambda_1| \leq 2\alpha_0^{2/3}|\Lambda_1|^2|\Lambda_2|, \quad (3.27)
\]
In the same way \( |\sigma_0'''| \leq 2\alpha_0^{2/3}|\Lambda_1|^2|\Lambda_2| \) and \( |\sigma_0''''| \leq 2\alpha_0^{2/3}|\Lambda_1|^2|\Lambda_2| \).
Take \( i \) and \( j \) such that \( i \notin \Omega_1, j \notin \Omega_2 \). Let \( g_{\lambda i}(s) = g_i(s), h_{\lambda j}(s) = h_j(s), \) and \( \Lambda_1 \times \mu_j = \Lambda_1^{(1)} \times \Lambda_2^{(1)}, \lambda_i \times \Lambda_2 = \Lambda_1^{(2)} \times \Lambda_2^{(2)} \). Let \( E_2^{(1)} = E_2 \cap \Lambda_2^{(1)}, E_2^{(2)} = E_2 \cap \Lambda_2^{(2)}, \beta_2^{(1)} = |E_2^{(1)}|/|\Lambda_2^{(1)}|, \) and \( \beta_2^{(2)} = |E_2^{(2)}|/|\Lambda_2^{(2)}| \). We have
\[
\sigma = \sigma_{i, j} = \sum_{s_1, s_2, r} h(s_1, s_2)g(s_1 + r, s_2 + r)f(s_1, s_2 + r) = \\
= \sum_{k, m} h(k, m)E_2^{(1)}(m) \sum_r g(k + r, m + r)f(k, m + r) \quad (3.28)
\]
Note that \( k \) in (3.29) belongs to \( \Lambda_1^{(2)} \). Using the Cauchy–Schwarz inequality, we obtain
\[
|\sigma|^2 \leq ||h||^2 \sum_{k, m} E_2^{(1)}(m) \sum_r g(k + r, m + r)f(k, m + r)^2 = \\
= ||h||^2 \sum_{k, m} E_2^{(1)}(m) \sum_{r, p} g(k + r, m + r)f(k, m + r)g(k + p, m + p)f(k, m + p) = \\
= ||h||^2 \sum_{k, m, u} g(k, m)g(k + u, m + u) \sum_r E_2^{(1)}(m - r)f(k - r, m)f(k - r, m + u) =
\]
\[ \|h\|_2^2 \sum_{k,m,u} g(k,m)g(k+u,m+u)E_1^{(2)}(k)E_1^{(2)}(k+u) \]
\[ \cdot \sum_r E_2^{(1)}(m-r)f(k-r,m)f(k-r,m+u). \]

We have \( k \in \Lambda_1^{(2)} \) and \( k-r \in \Lambda_1 \). It follows that \( r \in k - \Lambda_1 \in \Lambda_1^{(2)} - \Lambda_1 \). Since \( m-r \in \Lambda_2^{(1)} \) it follows that \( m \in \Lambda_2^{(1)} + r \in \Lambda_2^{(1)} + \Lambda_2^{(1)} - \Lambda_1 \). On the other hand \( k+u \in \Lambda_1^{(2)} \). Hence \( u \in \Lambda_2^{(1)} - \Lambda_1^{(2)} \) and \( m+u \in \Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1^{(2)} - \Lambda^{(2)} \). Let \( \Lambda_i = \Lambda' + \Lambda' + \Lambda' + \Lambda + \Lambda_i \) and \( m, m+u \in \Lambda_i \). Then \( m, m+u \in \Lambda_i \) and \( j = Q_{ij} = Q \). Using Lemma 2.3 for the Bohr set \( \Lambda \) and its \( \varepsilon_0 \)-attendant \( \Lambda' \), we obtain that the cardinality of \( \Lambda_i \) does not exceed \( 5|\Lambda_1| \). Using the Cauchy–Schwarz inequality, we get
\[
|\sigma|^4 \leq \|h\|_2^4 \left( \sum_k \sum_{m,u} g(k,m)g(k+u,m+u) \right) \tag{3.31}
\]
\[
\cdot \left( \sum_{k,m,u} E_2^{(1)}(k)E_1^{(2)}(k+u) \sum_{r,r'} E_1^{(2)}(m-r)E_1^{(2)}(m-r') \right) \times f(k-r,m)f(k-r,m+u)f(k-r',m)f(k-r',m+u). \]

Let \( \sigma^* = \sigma_{ij}^* = \sum_k \sum_{m,u} g(k,m)g(k+u,m+u) \). Let
\[
\Omega' = \{ s \in \Lambda_2 \mid |\delta_{\Lambda_1+s}(E_2) - \beta_2| \geq 4\alpha_0^{1/2} \text{ or} \right)
\[
\frac{1}{|\Lambda_1|} \sum_{n \in \Lambda_1 + s} |\delta_{\Lambda' + n}(E_2) - \beta_2|^2 \geq 4\alpha_0^{1/2} \}, \text{ and } G' = \Lambda_2 \setminus \Omega'. \]

By assumption \( \Lambda_1 \) is an \( \varepsilon_0 \)-attendant of \( \Lambda_2 \) and \( E_2 \) is an \( (\alpha_0, \varepsilon) \)-uniform subset of \( \Lambda_2 \). Using Lemma 3.3 we get \( |\Omega'| \leq 8\alpha_0^{1/2}|\Lambda_2| \). Let \( \Lambda = \Lambda' + \Lambda' + \Lambda' + \Lambda + \Lambda_1 \). Since \( \Lambda' \) is an \( \varepsilon_0 \)-attendant of \( \Lambda_1 \), it follows that for any \( s \in G' \) we have \( |\delta_{\Lambda_1+s}(E_2) - \beta_2| < 8\alpha_0^{1/2} \) and \( \sum_{n \in \Lambda_1 + s} |\delta_{\Lambda' + n}(E_2) - \beta_2|^2 < 8\alpha_0^{1/2} \Lambda_1 \). For an arbitrary \( i \in \Lambda_1 \) consider the set
\[
\Omega_i^* = \Omega_i^* = \{ j \in \Lambda_2 \mid |\delta_{\Lambda_1+j}(E_2) - \beta_2| \geq 8\alpha_0^{1/2} \} \text{ or} \right)
\[
\frac{1}{|\Lambda_1|} \sum_{n \in \Lambda_1 + j} |\delta_{\Lambda' + n}(E_2) - \beta_2|^2 \geq 8\alpha_0^{1/2} \} \tag{3.32}
\]

Since \( (\Lambda_2 \setminus \Omega_i^*) \supseteq (\Lambda_2 \setminus (G' - i)) \) it follows that \( \Omega_i^* \subseteq (\Lambda_2 \setminus (G' - i)) \). Since \( \Lambda_1 \) is an \( \varepsilon_0 \)-attendant of \( \Lambda_2 \), it follows that \( |\Lambda_2 \setminus (G' - i)| = |(\Lambda_2 + i) \setminus G' \| \geq |\Lambda_1 \setminus G' \| \geq (1 - 8\alpha_0^{1/2} - 8\kappa_0)|\Lambda_2|, \kappa_0 \leq \alpha_0^2 \). Hence \( |\Omega_i^*| \leq 8\alpha_0^{1/2}|\Lambda_2| + 8\kappa_0|\Lambda_2| \leq 16\alpha_0^{1/2}|\Lambda_2| \). This yields
\[
\frac{1}{|\Lambda_1|^2} \sum_{i \in \Omega_1, j \in \Omega_i^*} |\sigma_{ij}| \leq \frac{1}{|\Lambda_1|^2} \sum_{i \in \Omega_1} (16\alpha_0^{1/2}|\Lambda_2||\Lambda'|^2|\Lambda_1|) \leq 32\alpha_0^{1/2}|\Lambda_1|^2|\Lambda_2|. \tag{3.33}
\]

We have \( j \notin \Omega_2 \). Suppose in addition that \( j \notin \Omega_i^* \). Let \( \Omega_2^* = \Omega_2^*(i) = \Omega_2 \cup \Omega_i^* \).

**Lemma 3.4.** For any \( i \notin \Omega_1 \) and any \( j \notin \Omega_i^* \) the following holds we have either
\[
|\sigma_{ij}^*| \leq 16\delta_i^2 \beta_2^2 |\Lambda'|^2 |\Lambda_1|^2 |\Lambda_2|. \tag{3.34}
\]
or there exist two sets $F_1$, $F_2$ and a vector $\vec{y} = (y_1, y_2) \in G \times G$, $F_1 \subseteq E_1 \cap (\Lambda + y_1)$, $F_2 \subseteq E_2 \cap (\Lambda + y_2)$ such that \((3.22)\) and \((3.23)\) hold.

Note. Let $T$ be a subset of $G$, $|T| = \delta |G|$, $E_1 = E_2 = G$, $\beta_1 = \beta_2 = 1$ and let $g$ be the characteristic function of the set $A = \bigsqcup_{x \in G} \{x\} \times \{T + x\}$. Then it is easy to see that inequality \((3.34)\) is best possible in the case (up to constants). On the other hand \((3.22), (3.23)\) does not hold with $A$ equals $A$.

Proof. Let $\tilde{E}_2^{(2)} = E_2 \cap Q$ and $\tilde{E}_2^{(2)}(x) = \tilde{E}_2^{(2)}(-x)$. We have

$$
\sigma_{ij}^* = \sum_{k,m,u} g(k,m)g(k+u,m+u) \leq \sum_{k,m,u} g(k,m)E_1^{(2)}(k+u)\tilde{E}_2^{(2)}(m+u)
$$

$$
= \sum_{k,m} g(k,m)(E_1^{(2)}*\tilde{E}_2^{(2)})(k-m) = \sum_{k',m} g(k'+m,m)(E_1^{(2)}*\tilde{E}_2^{(2)})(k') \quad \text{.} (3.35)
$$

If $k'$ is fixed then the variable $m$ in \((3.35)\) belongs to the set of the cardinality $|\Lambda'|$. Recall that $|Q_{ij}| \leq 5|\Lambda_1|$. Lemma \((2.3)\) implies that $k'$ in the sum \((3.35)\) belongs to a set of cardinality at most $8|\Lambda_1|$. Since $i \notin \Omega_1$, it follows that the set $E_1^{(2)}$ is $\alpha_0$-uniform. Using Corollary \((3.1)\) we get

$$
\sigma_{ij}^* \leq \beta_1^{(2)} \sum_{k',m} g(k'+m,m)(\lambda_i*\tilde{E}_2^{(2)})(k') + 16\alpha_0^{2/3}|\Lambda'|^2|\Lambda_1|.
$$

We have $j \notin \Omega_1$. Hence

$$
\sigma_{ij}^* \leq \beta_1^{(2)} \sum_{k',m} g(k'+m,m)(\lambda_i*E_2)(k') + 16\alpha_0^{2/3}|\Lambda'|^2|\Lambda_1|
$$

$$
\leq \beta_1^{(2)}|\Lambda'| \sum_{k,m} g(k,m) + 32\alpha_0^{1/6}|\Lambda'|^2|\Lambda_1| \quad \text{.} (3.36)
$$

Suppose that $\sigma_{ij}^* > 16\beta_1^{(2)}|\Lambda'|^2|\Lambda_1|$. Since $i \notin \Omega_1$, it follows that $\beta_1/2 \leq \beta_1^{(2)} \leq 2\beta_1$. Using this and \((3.36)\), we get

$$
\sum_{k,m} g(k,m) \geq 8\beta_1^2|\Lambda'|^2|\Lambda_1| \quad \text{.} (3.37)
$$

Recall that $m$ belongs to the set $\tilde{\Lambda}_i + j$ in \((3.37)\). By Lemma \((2.3)\) we find

$$
\sum_{k,m} A(k,m)\Lambda'(k-i)\Lambda_1(m-i-j) \geq 4\beta_1^2|\Lambda'|^2|\Lambda_1| \quad \text{.} (3.38)
$$

We have $i \notin \Omega_1$ and $j \notin \Omega_1$. Using this fact, inequality \((3.38)\) and simple average arguments it is easy to see that there is a vector $\vec{y} = (y_1, y_2) \in G \times G$ and two sets $F_1 \subseteq E_1 \cap (\Lambda' + y_1)$, $F_2 \subseteq E_2 \cap (\Lambda' + y_2)$ such that \((3.22), (3.23)\) hold. This completes the proof of the lemma.

We have

$$
|\sigma|^4 \leq ||h||_2^4 \cdot \sigma^* \sum_{m,u} \sum_{r,r'} f(r,m)f(r,u)f(r',m)f(r',u) \quad \text{.} (3.39)
$$

$$
\sum_k E_1^{(2)}(k)E_1^{(2)}(k - m + u)E_2^{(1)}(m - k + r)E_2^{(1)}(m - k + r') = \quad \text{.} (3.40)
$$
We have $r \in \Lambda_1$ and $k + r \in \Lambda_2$. It follows that $k \in \Lambda_2^1 - \Lambda_1$. On the other hand $m - k \in \Lambda_2^1$. Hence $m \in \Lambda_2^1 + k \in \Lambda_2^1 + \Lambda_2^1 - \Lambda_1$. By symmetry $u$ belongs to $\Lambda_2 + \Lambda_2 - \Lambda_1$. Using Lemma 2.3 for $\Lambda_1$ and its $\varepsilon_0$-attendant $\Lambda'$, we obtain that $k$ and $m,u$ belongs to some translations of Bohr sets $W_1 = \Lambda_1^+$ and $W_2 = \Lambda_2^+$, respectively, and the cardinalities of these sets do not exceed $3|\Lambda_1|$. 

If $k$ is fixed, then $m,u,r,r'$ in (3.42) run some sets of the cardinalities at most $|\Lambda'|$.

Let $\Phi^1_{r,r'}(m) = f(r, -m)f(r', -m)W_2(m - i - j)$, $\Phi^2_{r,r'}(u) = f(r, -u)f(r', -u)W_2(u - i - j)$, $\Phi^m_{m,u}(r) = f(-r, m)f(-r, u)$, and $\Phi^4_{m,u}(r') = f(r', m)f(r', u)$. Consider the sets

$$B_1 = \{ k \mid |(\Phi^1_{r,r'} \ast E_1^2)(-k) - \beta^2_1(\Phi^1_{r,r'} \ast \Lambda^1_2)(-k)| \geq \alpha_0^{2/3}|\Lambda'| \}$$

$$B_2 = \{ k \mid |(\Phi^2_{r,r'} \ast E_1^2)(-k) - \beta^2_1(\Phi^2_{r,r'} \ast \Lambda^1_2)(-k)| \geq \alpha_0^{2/3}|\Lambda'| \}$$

$$B_3 = \{ k \in \Lambda_1 \mid |(\Phi^m_{m,u} \ast E_2^1)(k) - \beta^1_2(\Phi^m_{m,u} \ast \Lambda^1_2)(k)| \geq \alpha_0^{2/3}|\Lambda'| \}$$

$$B_4 = \{ k \in \Lambda_1 \mid |(\Phi^4_{m,u} \ast E_2^1)(k) - \beta^1_2(\Phi^4_{m,u} \ast \Lambda^1_2)(k)| \geq \alpha_0^{2/3}|\Lambda'| \}.$$ 

We have $i \notin \Omega_1$, $j \notin \Omega_2$. Using Corollary 3.4 we get $|B_1|, |B_2| \leq 3\alpha_0^{2/3}|\Lambda_1|$ and $|B_3|, |B_4| \leq \alpha_0^{2/3}|\Lambda_1|$. Let $B = B_1 \cup B_2 \cup B_3 \cup B_4$. Then $|B| \leq 8\alpha_0^{2/3}|\Lambda_1|$. Split $\sigma'$ as

$$\sigma' = \sum_{k \in B} \sum_{r,r'} E_1^2(k + r)E_2^1(k + r') \sum_m E_1^2(m - k)f(r, m)f(r', m)^2 +$$

$$+ \sum_{k \notin B} \sum_{r,r'} E_1^2(k + r)E_2^1(k + r') \sum_m E_1^2(m - k)f(r, m)f(r', m)^2 = \sigma_1 + \sigma_2$$

Let us estimate $\sigma_1$. Since $|B| \leq 8\alpha_0^{2/3}|\Lambda_1|$, it follows that

$$|\sigma_1| \leq 8\alpha_0^{2/3}|\Lambda'|^4|\Lambda_1|.$$ 

(3.44)

If $k \notin B$, then $k \notin B_1$. This implies that

$$\sigma_2 = \sum_{k \notin B} \sum_u \sum_{r,r'} f(r, u)f(r', u)E_1^2(u - k)E_2^1(k + r)E_2^1(k + r') \cdot$$

$$\sum_m f(r, m)f(r', m)E_1^2(m - k) =$$

$$= \sum_{k \notin B} \sum_u \sum_{r,r'} f(r, u)f(r', u)E_1^2(u - k)E_2^1(k + r)E_2^1(k + r')(\Phi^1_{r,r'} \ast E_2^2)(-k)$$
\[ = \beta_1^{(2)} \sum_{k \notin B} \sum_u \sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r') \cdot \]
\[ \sum_m f(r,m)f(r',m)\Lambda_1^{(2)}(m-k) + \]
\[ + \theta \alpha_0^{2/3} |\Lambda' | \sum_{k \notin B} \sum_u \sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r') \]
\[ = \beta_1^{(2)} \sum_{k \notin B} \sum_u \sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r') \cdot \]
\[ \sum_m f(r,m)f(r',m)\Lambda_1^{(2)}(m-k) + 4\theta \alpha_0^{2/3} |\Lambda' | \Lambda_1 , \quad (3.45) \]

where \( |\theta| \leq 1 \). Using these arguments for \( B_2, B_3 \) and \( B_4 \), we get
\[ |\sigma_2| \leq (\beta_1^{(2)})^2 (\beta_2^{(1)})^2 \sum_{m,u} \sum_{r,r'} f(r,m)f(r,u)f(r',m)f(r',u) \cdot \]
\[ \sum_k \Lambda_1^{(2)}(m-k)\Lambda_1^{(2)}(u-k)\Lambda_2^{(1)}(k+r)\Lambda_2^{(1)}(k+r') + 16\alpha_0^{2/3} |\Lambda' | \Lambda_1 , \quad (3.46) \]

It follows that
\[ |\sigma'| \leq |\sigma_1| + |\sigma_2| \leq (\beta_1^{(2)})^2 (\beta_2^{(1)})^2 \sum_{m,u} \sum_{r,r'} f(r,m)f(r,u)f(r',m)f(r',u) \cdot \]
\[ \sum_k \Lambda_1^{(2)}(m-k)\Lambda_1^{(2)}(u-k)\Lambda_2^{(1)}(k+r)\Lambda_2^{(1)}(k+r') + 32\alpha_0^{2/3} |\Lambda' | \Lambda_1 . \quad (3.47) \]

Using (3.42), we obtain
\[ |\sigma|^4 \leq \|h\|_2^4 \cdot \sigma^* \cdot (\beta_1^{(2)})^2 (\beta_2^{(1)})^2 \sum_k \sum_{r,r'} \Lambda_1^{(1)}(k+r)\Lambda_2^{(1)}(k+r') \cdot \]
\[ \left| \sum_m \Lambda_1^{(2)}(m-k)f(r,m)f(r',m) \right|^2 + 32\|h\|_2^2 \cdot \sigma^* \cdot \alpha_0^{2/3} |\Lambda' | \Lambda_1 , \quad (3.48) \]

Since \( i \notin \Omega_1, j \notin \Omega_2 \), it follows that \( \beta_1^{(2)} \leq 2\beta_1 \) and \( \beta_2^{(1)} \leq 2\beta_2 \). Whence
\[ |\sigma_{ij}|^4 \leq 4^2 \beta_1^2 \beta_2^2 \cdot \|h\|_2^4 \cdot \sigma_{ij}^* \cdot \sum_k \sum_{r,r'} \Lambda_2^{(1)}(k+r)\Lambda_2^{(1)}(k+r') \cdot \]
\[ \left| \sum_m \Lambda_1^{(2)}(m-k)f(r,m)f(r',m) \right|^2 + 2^5 \alpha_0^{2/3} \cdot \|h\|_2^4 \cdot \sigma_{ij}^* \cdot |\Lambda' | \Lambda_1 . \quad (3.49) \]

Let \( \alpha_{ij} = \sum_k \sum_{r,r'} \Lambda_2^{(1)}(k+r)\Lambda_2^{(1)}(k+r') \cdot \sum_m \Lambda_1^{(2)}(m-k)f(r,m)f(r',m)^2 \).

Suppose that there are \( i \notin \Omega_1, j \notin \Omega_2 \) such that \( \|h_j\|_2^2 \geq 8\delta \beta_1 \beta_2 |\Lambda' | \Lambda_1 \). It follows that
\[ \sum_{k,m} \Lambda(k,m)\Lambda'(m-j) \geq 8\delta \beta_1 \beta_2 |\Lambda' | \Lambda_1 \cdot \quad (3.50) \]

Let \( F'_1 = E_1, F'_2 = E_2^{(1)} \). We have \( j \notin \Omega_2(i) \). Using this and (3.50), we get
\[ \left| A \cap F'_1 \times F'_2 \right| \geq 4\delta \beta_1 \beta_2 |F'_1||F'_2| \]
and

\[ |F'_1| = \beta_1|A_1|, \quad |F'_2| \geq 2^{-1} \beta_2|A'|. \]

Using simple average arguments it is easy to see that there are a vector \( \bar{y} = (y_1, y_2) \in G \times G \) and two sets \( F_1 \subseteq E_1 \cap (A' + y_1), F_2 \subseteq E_2 \cap (A' + y_2) \) such that (3.22), (3.23) hold.

Using Lemma 3.4, we obtain

\[
\sum_{i \in \Lambda_1, j \in \Lambda_2} |\sigma_{ij}| \leq 8 \delta^{3/4} \beta_1^3 \beta_2^{3/2} |A'| \beta_2^{1/2} |\Lambda_1| |A'_1|^{1/2} + 4 \alpha_0^{1/6} |A'|^2 |A_1|^2 |A_2|.
\]

By assumption the function \( f \) is rectilinearly \((\alpha, \varepsilon)-\)uniform. Clearly,

\[
\sum_{i \in \Lambda_1, j \in \Lambda_2} \alpha_{ij} = \sum_{i \in \Lambda_1, j \in \Lambda_2} \sum_k \sum_{r, r'} \mu_j(k+r) \mu_j(k+r') \cdot \left| \sum_m \lambda_i(m-k) f(r, m) f(r', m) \right|^2.
\]

It follows that

\[
\sum_{i \notin \Omega_1, j \notin \Omega_2} |\sigma_{ij}| \leq 8 \alpha_1^{1/4} \delta^{3/4} \beta_1^3 \beta_2^{3/2} |A'_1| |A_1|^2 |A_2|^2 + 4 \alpha_0^{1/6} |A'|^2 |A_1|^2 |A_2|^2 + 8 \alpha_1^{1/4} \delta^{3/4} \beta_1^3 \beta_2^{3/2} |A'_1|^2 |A_2|^2 \leq 16 \alpha_1^{1/4} \delta^{3/4} \beta_1^3 \beta_2^{3/2} |A_1|^4 |A_2|^2,
\]

as required.

The next result is the main in this section.

Let \( \Lambda_1, \Lambda_2 \) be Bohr sets, \( \Lambda_1 \subseteq \Lambda_2, \Lambda_1 = \Lambda(S, \varepsilon_1), S \subseteq \tilde{G} \) and let \( E_1 \subseteq \Lambda_1, E_2 \subseteq \Lambda_2, |E_1| = \beta_1|A_1|, |E_2| = \beta_2|A_2| \). By \( \mathcal{P} \) denote the product set \( E_1 \times E_2 \).

**Theorem 3.2.** Let \( A \) be an arbitrary subset of \( E_1 \times E_2 \) of cardinality \( \delta|E_1||E_2| \). Suppose that the sets \( E_1, E_2 \) are \((\alpha_0, 2^{-10} \varepsilon^2)-\)uniform, \( \alpha_0 = 2^{-2000} \delta^{16} \beta_1^3 \beta_2^3, \varepsilon = (2^{-100} \alpha_0^2)/(100d) \). Let \( A \) be rectilinearly \((\alpha, \alpha_1, \varepsilon)-\)uniform, \( \alpha = 2^{-100} \delta^2, \alpha_1 = 2^{-7}, \) and

\[
\log N \geq 2^{10} \log \frac{1}{\varepsilon_1 \varepsilon}.
\]

Then either \( A \) contains a triple \( \{(k, m), (k + d, m), (k, m + d)\} \), where \( d \neq 0 \) or there exists a Bohr set \( \Lambda \), two sets \( F_1, F_2 \) and a vector \( \bar{y} = (y_1, y_2) \in G \times G, F_1 \subseteq E_1 \cap (\Lambda + y_1), F_2 \subseteq E_2 \cap (\Lambda + y_2) \) such that \( \Lambda \) is an \( 2^{-16} \)–attendant of \( \Lambda_1 \) and

\[
|F_1| \geq 2^{-20} \beta_1|\Lambda|, \quad |F_2| \geq 2^{-20} \beta_2|\Lambda| \quad \text{and} \quad \delta_{F_1 \times F_2}(A) \geq \frac{3}{2} \delta.
\]
Proof. Let $\Lambda'$ be an $\varepsilon$-attendant set of $\Lambda_1$ to be chosen later, and $\lambda_i = \Lambda' + i$, $i \in \Lambda_1$. Let $G_i = (\lambda_i \times \Lambda_2) \cap A$, $f_i(\tilde{s}) = f(s_1 + i, s_2)\Lambda'(s_1, s_2)$, $i \in \Lambda_1$. By $G_i$ denote the characteristic functions of the sets $G_i$. Let

$$B_1 = \{i \in \Lambda_1 \mid E_1 \cap \lambda_i \text{ is not } (8a_0^{1/4}, \varepsilon)\text{-uniform}\},$$

$$B_2 = \{i \in \Lambda_1 \mid |\delta_{\lambda_i}(E_1) - \beta_1| \geq 4a_0^{1/2}\},$$

$$B_3 = \{i \in \Lambda_1 \mid \|f_i\|_{L_A^4(\Lambda' \times \Lambda_2, \varepsilon)} > \alpha\beta_1^2\beta_2^2|\Lambda'|^4|\Lambda_2^2|\},$$

and $B = B_1 \cup B_2 \cup B_3$.

By assumption $E_1$ is $(\alpha_0, \varepsilon)$-uniform. By Lemma 3.3 we get $|B_1| \leq 8a_0^{1/4}|\Lambda_1|$, and $|B_2| \leq 8a_0^{1/4}|\Lambda_1|$. Since $A$ is rectilinearly $(\alpha, \alpha_1, \varepsilon)$-uniform, it follows that $|B_3| \leq \alpha_1|\Lambda_1|$. Hence $|B| \leq 16a_0^{1/4}|\Lambda_1| + \alpha_1|\Lambda_1| \leq 2\alpha_1|\Lambda_1|$. Using Lemma 2.3 we obtain

$$A(\tilde{s}) = \frac{1}{|\Lambda'|} \sum_{i \in \Lambda_1} G_i(\tilde{s}) + \epsilon(\tilde{s}),$$

where $\|\epsilon\|_1 \leq 2\kappa|\Lambda_1||\Lambda_2|$, $\kappa = \alpha^2_0$. Consider the sum

$$\sigma = \frac{1}{|\Lambda'|} \sum_{i \in \Lambda_1} \sum_{x,y} G_i(x + y, y).$$

We have $|A| = \delta\beta_1\beta_2|\Lambda_1||\Lambda_2|$. Using (3.55), we get

$$\sigma \geq \frac{7\delta\beta_1\beta_2}{8} |\Lambda_1||\Lambda_2|. \quad (3.57)$$

Split $\sigma$ as

$$\sigma = \frac{1}{|\Lambda'|} \sum_{i \in B} \sum_{x,y} G_i(x + y, y) + \frac{1}{|\Lambda'|} \sum_{i \notin B} \sum_{x,y} G_i(x + y, y) = \sigma_1 + \sigma_2. \quad (3.58)$$

Let us estimate $\sigma_1$. We have

$$\sigma_1 = \frac{1}{|\Lambda'|} \sum_{i \in B_1 \setminus (B_1 \cup B_2)} \sum_{x,y} G_i(x + y, y) + \frac{1}{|\Lambda'|} \sum_{i \in B_1 \cup B_2} \sum_{x,y} G_i(x + y, y) \leq \quad (3.59)$$

$$\leq \frac{1}{|\Lambda'|} \sum_{i \in B_3 \setminus (B_1 \cup B_2)} \sum_{x,y} G_i(x + y, y) + 16a_0^{1/4}|\Lambda_1||\Lambda_2|. \quad (3.60)$$

Suppose that there exists $i \notin B_1 \cup B_2$ such that

$$\sum_{x,y} G_i(x + y, y) \geq 4\delta\beta_1\beta_2|\Lambda'|||\Lambda_2|. \quad (3.61)$$

In other words

$$\sum_{x,y} G_i(x, y) \geq 4\delta\beta_1\beta_2|\Lambda'||\Lambda_2|. \quad (3.62)$$

Put $y_1 = i$, $y_2 = 0$ and $F_1 = (\Lambda' + i) \cap E_1$. Since $i \notin B_2$, it follows that $|F_1| \geq \beta_1|\Lambda'|/2$. Using simple average arguments we see that there exists an element $a$ such that $F_2 = (\Lambda' + a) \cap E_2$ has the cardinality at least $\beta_2|\Lambda_1|/2$ and for $\tilde{y} = (i, a)$ we have

$$|A \cap (F_1 \times F_2)| > 2\delta|F_1||F_2|. \quad (3.63)$$

Thus we get (3.55), (3.56) and the theorem is proven in the case.
ON A TWO–DIMENSIONAL ANALOG OF SZEMERÉDI’S THEOREM IN ABELIAN GROUPS

We have $\sigma_1 = 2^{-7}$. Using $|B_3| \leq \sigma_1|A|$ and $\sigma_0^{1/4} \leq 2^{-4} \sigma_1 \beta_1 \beta_2$, we obtain

$$
\sigma_1 \leq 4\delta \beta_1 \beta_2 |A'||B_3||A_2| + 16\alpha_0^{1/4}|A_1||A_2| \leq 2^{-3} \delta \beta_1 \beta_2 |A'||A_1||A_2|.
$$

(3.61)

Using this and (3.61), (3.63), we obtain

$$
\frac{1}{|\Lambda'|} \sum_{i \notin B} \sum_{x,y} G_i(x + y, y) \geq \frac{3\delta \beta_1 \beta_2}{4} |A_1||A_2|.
$$

(3.62)

The formula (3.62) implies that there exists $i_0 \notin B$ such that

$$
\sum_{x,y} G_{i_0}(x + y, y) \geq \frac{3}{4} \delta \beta_1 \beta_2 |A'||A_2|.
$$

(3.63)

Let $G'(\vec{s}) = G_{i_0}(\vec{s})$. We have

$$
\sum_k \sum_m G'(k + m, m) \geq 2^{-3} \delta \beta_1 \beta_2 |A'||A_2|.
$$

(3.64)

We have $m \in A_2$ and $k + m \in \Lambda_i$. It follows that $k \in \lambda_i - A_2$. Using Lemma 2.23 we obtain that $k$ belongs to a set of cardinality at most $2|A_2|$. By the Cauchy–Schwartz inequality, we get

$$
2^{-6} \delta^2 \beta_1^2 \beta_2^2 |A'||A_2|^2 \leq \sum_k \left( \sum_m G'(k + m, m) \right)^2 \cdot 2|A_2|.
$$

(3.65)

It follows that

$$
\sum_k \left( \sum_m G'(k + m, m) \right)^2 = \sum_k \sum_{m,p} G'(k + m, m) G'(k + p, p) \geq 2^{-7} \delta^2 \beta_1^2 \beta_2^2 |A'||A_2|.
$$

(3.66)

Consider the sum

$$
\sigma_0 = \sum_{s_1, s_2, r} G'(s_1, s_2) G'(s_1 + r, s_2 + r) A(s_1, s_2 + r).
$$

(3.67)

We have

$$
G'(s_1, s_2) G'(s_1 + r, s_2 + r) f(s_1, s_2 + r) =
$$

$$
G'(s_1, s_2) G'(s_1 + r, s_2 + r) f_{i_0}(s_1, s_2 + r),
$$

(3.68)

where $f_{i_0}$ is the restriction of the function $f$ to $G'$. It follows that

$$
\sigma_0 = \delta \sum_{s_1, s_2, r} G'(s_1, s_2) G'(s_1 + r, s_2 + r) \mathcal{P}(s_1, s_2 + r) +
$$

$$+
\sum_{s_1, s_2, r} G'(s_1, s_2) G'(s_1 + r, s_2 + r) f(s_1, s_2 + r) =
$$

$$
= \delta \sum_{s_1, s_2, r} G'(s_1, s_2) G'(s_1 + r, s_2 + r) + \sum_{s_1, s_2, r} G'(s_1, s_2) G'(s_1 + r, s_2 + r) f_{i_0}(s_1, s_2 + r).
$$

(3.69)

The inequality (3.66) implies that the first term in (3.69) is greater than $2^{-7} \delta^3 \beta_1^2 \beta_2^2 |A'||A_2|$. Since $i_0 \notin B$, it follows that $||f_{i_0}||^4 \leq \alpha \beta_1^2 \beta_2^2 |A'||A_2|$ and $\delta_{\lambda_{i_0}}(E_1) \leq 2\beta_1$. By assumption $\alpha = 2^{-100} \delta^9$. Using Theorem 3.1 and (3.66), we obtain that either the second term in (3.69) does not exceed

$$
2^{10} \alpha^{1/4} \delta^3 \beta_1^2 \beta_2^2 |A'||A_2| \leq 2^{-8} \delta^3 \beta_1^2 \beta_2^2 |A'||A_2|.
$$


or there is a vector $\vec{y} = (y_1, y_2) \in G \times G$ and two sets $F_1 \subseteq E_1 \cap (\hat{A} + y_1)$, $F_2 \subseteq E_2 \cap (\hat{A} + y_2)$ such that (3.53), (3.54) hold. If we have the second situation then we are done and $\hat{A}$ is an $\varepsilon$-attendant of $\Lambda'$. In the other case $\sigma_0 \geq 2^{-7} \delta^3 \beta_1^2 \beta_2^2 |\Lambda'|^2 |A_2|$.

The sum (3.67) is the number of triples $\{(k, m), (k + d, m), (k, m + d)\}$, where $k \in \Lambda_{i_0}$, $m \in \Lambda_2$, $d \in G$. The number of triples with $d = 0$ does not exceed $|\Lambda'||A_2|$. By assumption $\log N \geq 2^{10} d \log \frac{1}{\varepsilon}$. Using Lemma 2.1 we get $|\Lambda'| > 2^8 (\delta^3 \beta_1^2 \beta_2^2)^{-1}$. Hence, $2^{-8} \delta^3 \beta_1^2 \beta_2^2 |\Lambda'|^2 |A_2| > |\Lambda'||A_2|$. It follows that $A$ contains a triple $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \neq 0$. This completes the proof.


**Lemma 4.1.** Let $\Lambda_1$, $\Lambda_2$ be Bohr sets, $\Lambda_1 \leq \Lambda_2$, and $\Lambda'$ be an $\varepsilon$-attendant set of $\Lambda_1$, $\varepsilon = \kappa/(100d)$. Let set $A$ be a subset of $C \subseteq \Lambda_1 \times \Lambda_2$ of cardinality $\delta |C|$. By $B$ define the set of $s \in \Lambda_1$ such that $|A \cap ((\Lambda' + s) \times \Lambda_2)| < (\delta - \eta) |C \cap ((\Lambda' + s) \times \Lambda_2)|$, where $\eta > 0$. Then

$$\sum_{s \in (\Lambda_1 \setminus B)} |A \cap ((\Lambda' + s) \times \Lambda_2)| \geq \delta \sum_{s \in (\Lambda_1 \setminus B)} |C \cap ((\Lambda' + s) \times \Lambda_2)| + n \sum_{s \in B} |C \cap ((\Lambda' + s) \times \Lambda_2)| - 4\kappa |\Lambda'||A_1||A_2| .$$

**Proof.** Using Lemma 2.3 we get

$$\delta |C| = \sum_{\vec{s}} A(\vec{s}) \Lambda_1(k) \Lambda_2(m) = \frac{1}{|\Lambda|} \sum_{n \in \Lambda_1} \sum_{\vec{s}} A(\vec{s}) ((\Lambda' + n) \times \Lambda_2)(\vec{s}) + 2\theta \kappa |\Lambda_1||A_2| ,$$

where $|\theta| \leq 1$. Split the sum (4.1) into a sum over $n \in B$ and a sum over $n \in \Lambda_1 \setminus B$. We have

$$\delta |C| < \frac{1}{|\Lambda|} (\delta - \eta) \sum_{n \in B} |C \cap ((\Lambda' + n) \times \Lambda_2)| +$$

$$+ \frac{1}{|\Lambda|} \sum_{n \in (\Lambda_1 \setminus B)} |A \cap ((\Lambda' + n) \times \Lambda_2)| + 2\kappa |\Lambda_1||A_2| .$$

In the same way

$$|C| = \frac{1}{|\Lambda|} \sum_{n \in B} |C \cap ((\Lambda' + n) \times \Lambda_2)| + \frac{1}{|\Lambda|} \sum_{n \in (\Lambda_1 \setminus B)} |C \cap ((\Lambda' + n) \times \Lambda_2)| + 2\theta_1 \kappa |\Lambda_1||A_2| ,$$

where $|\theta_1| \leq 1$. Combining (4.2) and (4.3), we obtain the required result.

Let $X$ be a finite set, $\mu$ be a measure on $X$ and let $Z : X \to \mathbb{R}$ be a function. By $\mathbb{E}Z'$ denote the sum $\frac{1}{|X|} \sum_{x \in X} Z(x)$. The following lemma is well-known (see e.g. [18]).

**Lemma 4.2.** Let $p$ be a real number. Suppose that $Z : X \to [-1, 1]$ is a function
such that $E Z = 0$ and $E |Z|^p = \sigma^p$. Then

$$
\mu \left\{ x \in X : Z > \frac{\sigma^p}{5} \right\} \geq \frac{\sigma^p}{5},
$$

(4.4)

**Proof.** Suppose that (1.3) does not hold. Since $E Z = 0$ it follows that

$$
-E Z 1_{\{Z < 0\}} = E Z 1_{\{Z > 0\}} \leq \mu \{ x : Z > 5^{-1} \sigma^p \} + E Z 1_{\{0 < Z \leq 5^{-1} \sigma^p \}} \leq \frac{2}{5} \sigma^p,
$$

where $1_{\{Z < 0\}}, 1_{\{Z > 0\}}$ are the characteristics functions of the sets $\{ x : Z(x) < 0 \}$, $\{ x : Z(x) > 0 \}$ respectively. We have $|Z(x)| \leq 1$ for all $x \in X$. Hence

$$
\sigma^p = E |Z|^p = E |Z|^p 1_{\{Z < 0\}} + E |Z|^p 1_{\{Z > 0\}} \leq 2 E Z 1_{\{Z > 0\}} \leq \frac{4}{5} \sigma^p
$$

(4.5)

with contradiction. \[\square\]

We need in the proposition concerning the properties of not rectilinearly $\alpha$–uniform sets. The similar proposition was proven in [24, 26, 15, 18].

**Proposition 4.1.** Let $A$ be a subset of $E_1 \times E_2$ of cardinality $|A| = \delta |E_1| |E_2|$. Suppose that $\alpha > 0$ is a real number, $\alpha \leq \delta^4/8$, and $A$ is not rectilinearly $\alpha$–uniform. Then there are two sets $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$ such that

$$
|A \cap (F_1 \times F_2)| > (\delta + 2^{-15} \cdot \min\{\alpha^2 \delta^{-5}, \alpha \delta^{-2}\}) |F_1| |F_2|
$$

and

$$
|F_1| \geq 2^{-15} \min\{\alpha^2 \delta^{-5}, \alpha \delta^{-2}\} |E_1|, \quad |F_2| \geq 2^{-15} \min\{\alpha^2 \delta^{-5}, \alpha \delta^{-2}\} |E_2|.
$$

(4.6)

(4.7)

**Proof.** Denote by $f$ the balanced function of $A$. Suppose that

$$
\sum_x \left| \sum_y f(x, y) \right|^2 \leq \alpha \delta^{-2} |E_1|^2 |E_2|^2/16
$$

(4.8)

and

$$
\sum_y \left| \sum_x f(x, y) \right|^2 \leq \alpha \delta^{-2} |E_1|^2 |E_2|^2/16.
$$

(4.9)

If (4.8) or (4.9) is not true then we can use Lemma 4.2 and find two sets $F_1, F_2$ such that (4.10), (1.4) hold. Let us prove that

$$
\|A\|^4 \geq (\delta^4 + \alpha/2) |E_1|^2 |E_2|^2
$$

(4.10)

By assumption $\|f\|^4 \geq \alpha |E_1|^2 |E_2|^2$. Using the obvious formulas $A = f + \delta (E_1 \times E_2)$ and $\sum_{x, y} f(x, y) = 0$, we get

$$
\|A\|^4 \geq (\delta^4 + \alpha) |E_1|^2 |E_2|^2 + \delta \sum_{x, x', y, y'} f(x, y) f(x', y) f(x, y') + \delta \sum_{x, x', y, y'} f(x, y) f(x', y) f(x, y')
$$

(4.11)

$$
+ \delta \sum_{x, x', y, y'} f(x, y) f(x', y') f(x, y') + \delta \sum_{x, x', y, y'} f(x', y) f(x, y') f(x', y')
$$

(4.12)

$$
+ \delta \sum_{x, x', y, y'} f(x, y) f(x, y') f(x', y') + \delta \sum_{x, x', y, y'} f(x', y) f(x, y') f(x', y')
$$

(4.13)
Let \( \tilde{\alpha} \) hold.

Let us prove that any term in (4.12) — (4.13) at most \( \tilde{\alpha} \). We have seen that the sum of four terms in (4.15) — (4.16) does not exceed \( \tilde{\alpha} \).

It is easy to see that two summands in (4.14) equal zero. Using (4.8) and (4.9), we find two sets \( A \) such that (4.6), (4.7) hold. So any term in (4.12) — (4.13) at most \( \alpha/(16\delta) \). Without loss of generality it can be assumed that the first summand in (4.12) is greater than \( \alpha/(16\delta) \). We have

\[
\frac{\alpha}{16\delta} \leq \left( \sum_{x,y} |f(x,y)|^3 \right)^{1/3} \cdot \left( \sum_{x,y} \sum_{x'} |f(x',y)|^{3/2} \cdot \sum_{y'} |f(x,y')|^{3/2} \right)^{2/3} \leq 2\delta^{1/3} \cdot \left( \sum_y \sum_{x'} |f(x',y)|^{3/2} \cdot \sum_x |f(x,y')|^{3/2} \right)^{2/3}
\]

Thus, we have for example

\[
\sum_y \sum_{x'} |f(x',y)|^{3/2} \geq \frac{\alpha^{3/4}}{16\delta} \geq \frac{\alpha^2}{16\delta^3}.
\]

Using Lemma 4.2 and find two sets \( F_1, F_2 \) such that (4.9), (4.10) hold. So any term in (4.12) — (4.13) does not exceed \( \alpha/(16\delta) \) and we have proved (4.10).

Let \( e(x,y) = \{(\tilde{x},y) \in A \mid (\tilde{x},y) \in A \) and \( (x,y) \in A \} \) and \( N_x = \{y \mid (x,y) \in A \}, N_y = \{x \mid (x,y) \in A \} \). Clearly,

\[
\|A\|^4 = \sum_{(x,y)\in A} e(x,y).
\]

Let \( \tilde{X} = \{x \in E_1 : |\sum_y f(x,y)| \leq \alpha|E_2|/(32\delta^3)\} \) and \( \tilde{Y} = \{y \in E_2 : |\sum_x f(x,y)| \leq \alpha|E_1|/(32\delta^3)\} \). Let also \( X^c = E_1 \setminus \tilde{X} \) and \( Y^c = E_2 \setminus \tilde{Y} \). Note that \( |X^c| \leq \zeta |E_1| \), \( |Y^c| \leq \zeta |E_2| \), where \( \zeta = \alpha/(128\delta^2) \). Indeed, if \( |X^c| > \zeta |E_1| \) then \( \sum_x |\sum_y f(x,y)| \geq \alpha^2 |E_1| |E_2|/(2^{12}\delta^3) \). Using Lemma 4.2 and find two sets \( F_1, F_2 \) such that (4.9), (4.10) hold.

Let us prove that

\[
\sum_{x \in \tilde{X}, y \in \tilde{Y}} A(x,y)e(x,y) \geq (\delta^4 + \alpha/4) |E_1|^2 |E_2|^2.
\]

We have

\[
\|A\|^4 = \sum_{x \in \tilde{X}, y \in \tilde{Y}} A(x,y)A(x',y)A(x',y)A(x',y') + \sum_{x \in \tilde{X}, y \in Y^c} A(x,y)A(x',y)A(x',y)A(x',y') + \sum_{x \in \tilde{X}, y \in Y^c} A(x,y)A(x',y)A(x',y)A(x',y') + \sum_{x \in \tilde{X}, y \in Y^c} A(x,y)A(x',y)A(x',y)A(x',y')
\]

We have

\[
\|A\|^4 = \sum_{x \in \tilde{X}, y \in \tilde{Y}} A(x,y)A(x',y)A(x',y)A(x',y') + \sum_{x \in \tilde{X}, y \in Y^c} A(x,y)A(x',y)A(x',y)A(x',y') +
\]
on a two–dimensional analog of Szemerédi’s theorem in abelian groups.

\[ + \sum_{x \in X', y \in Y^*} A(x, y)A(x', y')A(x, y')A(x', y') = \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3. \]

Clearly, \( \sigma_3 \leq |X'|^2|Y^*| \cdot \delta |E_1||E_2| \leq \alpha|E_1|^2|E_2|^2 / 16. \) Further, \( \sigma_1 \leq |Y^*| \sum_{x, y \in Y^*} \sum_{x', y' \in Y^*} |A(x, y)| |A(x', y')|^2 + |Y^*| \sum_{y' \in Y^*} \sum_{x} |A(x, y')|^2 \)

\[ \leq 4\delta^2|Y^*|^2|E_1|^2|E_2| + |Y^*|^2|E_1|^2 \leq \alpha|E_1|^2|E_2|^2 / 16. \]

In the same way \( \sigma_2 \leq \alpha|E_1|^2|E_2|^2 / 16. \) Using (4.10) and (4.17), we get (4.18).

By (4.18), we find \((x_0, y_0) \in \Lambda \cap (\tilde{X} \times \tilde{Y})\) such that

\[ e(x_0, y_0) \geq (\delta^3 + \frac{\alpha}{4\delta^3})|E_1||E_2|. \]  

(4.19)

Put \( F_1 = N_{y_0}, F_2 = N_{x_0}. \) By definition of \( \tilde{X}, \tilde{Y}, \) we get \( ||N_x - \delta|E_2|| \leq \alpha|E_2|/(32\delta^3) \) and \( ||N_{y_0} - \delta|E_1|| \leq \alpha|E_1|/(32\delta^3). \) In particular \( |F_1|, |F_2| \geq \delta^2 / 2 \) and (4.19) holds. Obviously, \( e(x, y) = ||N_y \times N_x \cap \Lambda||. \) Using (4.19) and \( \alpha \leq \delta^4 / 8, \) we obtain

\[ |A \cap (F_1 \times F_2)| \geq (\delta + \frac{\alpha}{4\delta^3})(1 - \frac{\alpha}{16\delta^3})|F_1||F_2| \geq (\delta + \frac{\alpha}{8\delta^3})|F_1||F_2|. \]

and we get (4.7). This concludes the proof.

Let \( \Lambda_1, \Lambda_2 \) be Bohr sets, \( \Lambda_1 \leq \Lambda_2, \Lambda_1 = \Lambda(S, \varepsilon_0), |S| = d, \) and \( E_1 \subseteq \Lambda_1, E_2 \subseteq \Lambda_2, |E_1| = \beta_1|\Lambda_1|, |E_2| = \beta_2|E_2|. \) Let \( P \) be a product set \( E_1 \times E_2. \)

**Theorem 4.1. Let \( A \) be a subset of \( P \) of cardinality \( |A| = \delta|E_1||E_2|. \) Suppose that \( A \) has no triples \( \{(k, m), (k + d, m), (k, m + d)\} \) with \( d \neq 0, E_1, E_2 \) are \((\alpha_0, 2^{-10}\varepsilon)^2\)-uniform, \( \alpha_0 = 2^{-2000}\delta^9 \beta_1^1 \beta_2^{18} \beta_3^{38}, \varepsilon = (2^{-100}\alpha_0^2)/(100d), \varepsilon' = 2^{-10}\varepsilon^2, \) and \( \log N \geq 2^{10}d \log \frac{1}{\varepsilon_0}. \)**

Then there exists a Bohr set \( \tilde{\Lambda}, \) two sets \( F_1, F_2 \) and a vector \( \tilde{y} = (y_1, y_2) \in G \times G, F_1 \subseteq E_1 \cap (\tilde{\Lambda} + y_1), F_2 \subseteq E_2 \cap (\tilde{\Lambda} + y_2) \) such that

\[ |F_1| \geq 2^{-500}\delta^{22} \beta_1|\tilde{\Lambda}|, \quad |F_2| \geq 2^{-500}\delta^{22} \beta_2|\tilde{\Lambda}| \quad \text{and} \]

\[ \delta_{F_1 \times F_2}(A) \geq \delta + 2^{-500}\delta^{22}. \]

(4.20) (4.21)

Besides that for \( \tilde{\Lambda} = \Lambda(\tilde{S}, \varepsilon) \) we have \( \tilde{S} = S \) and \( \varepsilon \geq 2^{-5}\varepsilon'\varepsilon_0. \)

**Proof.** Let \( \Lambda' \) be an \( \varepsilon \)-attendant of \( \Lambda_1, \) and \( \Lambda'' \) be an \( \varepsilon \)-attendant of \( \Lambda' \) to be chosen later. Suppose that \( A \) is rectilinearly \((\alpha, \alpha_1, \varepsilon)\)-uniform, \( \alpha = 2^{-100}\delta^9, \alpha_1 = 2^{-7}. \) Using Theorem 3.2 we obtain that either \( A \) contains a triple \( \{(k, m), (k + d, m), (k, m + d)\} \) with \( d \neq 0 \) or (4.20), (4.21) hold. At the first case we get a contradiction, at the second case we obtain the required result. Hence the set \( A \) is not rectilinearly \((\alpha, \alpha_1, \varepsilon)\)-uniform.
Let
\[ B_1 = \{ s \in \Lambda_1 \mid |\delta_{\Lambda'}(E_1) - \beta_1| \geq 4a_0^{1/2} \}, \]
\[ B_2 = \{ s \in \Lambda_1 \mid \Lambda' \cap (E_1 - s) \text{ is not } (8a_0^{1/4}, \varepsilon)\text{-uniform} \}, \]
and
\[ B = \{ i \in \Lambda_1 \mid \| f_i \|_{\Lambda' \times \Lambda_2, \varepsilon} > \alpha_0 \beta_i |\Lambda'| \|\Lambda_2\| \|A_2\| \}. \]

Since \( A \) is not rectilinearly \((\alpha, \alpha_1, \varepsilon')\)-uniform, it follows that \(|B| > \alpha_1 |\Lambda_1|\). By assumption \( E_1, E_2 \) are \((\alpha_0, \varepsilon')\)-uniform. Using Lemma 3.3, we obtain \(|B_1| \leq 4a_0^{1/2} |\Lambda_1|\), \(|B_2| \leq 8a_0^{1/2} |\Lambda_1|\). Let \( B_3 = B_1 \cup B_2\). Then \(|B_3| \leq 12a_0^{1/2} |\Lambda_1|\). Let \( B' = B \setminus B_3\).

Since \( 4a_0^{1/2} < \alpha_1 \), it follows that \(|B'| \geq \alpha_1 |\Lambda_1|/2\). Note that for all \( l \in B' \) we have
\[ |\delta_{\Lambda'}(E_1) - \beta_1| < 4a_0^{1/2}. \]  

Let \( \eta = 2^{-100} \alpha^{3/2} \). Let \( \lambda_l = \Lambda' + l, l \in \Lambda_1\). Suppose that for any \( l \in B' \) we have
\[ |A \cap (\lambda_l \times \Lambda_2)| \leq (\delta - \eta)|\lambda_l \cap E_1|\|A_2 \cap E_2\|. \]  

Let \( B'^c = \Lambda_1 \setminus B'\). Using Lemma 4.1 and (4.22), we get
\[
\sum_{l \in B'^c} |A \cap (\lambda_l \times \Lambda_2)| \geq \delta |\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + \eta |\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| - \alpha_0^3 |\Lambda'| |\Lambda_1| |\Lambda_2| \\
\geq \delta |\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + \eta \frac{\alpha_1 |\Lambda_1| \beta_1 |\Lambda'|}{2} |\lambda_l \cap E_1| \\
\geq \delta |\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + 2^{-3} \alpha_1 \eta \beta_1 |\Lambda'| |\Lambda_1| |\Lambda_2|. \]  

We have
\[
\sum_{l \in B_1} |A \cap (\lambda_l \times \Lambda_2)| \leq 4a_0^{1/2} |\Lambda_1| |\Lambda'| |\Lambda_2| \leq 2^{-4} \alpha_1 \eta \beta_1 |\Lambda'| |\Lambda_1| |\Lambda_2|. \]  

Combining (4.24) and (4.25), we obtain
\[
\sum_{l \in (B'^c \setminus B_1)} |A \cap (\lambda_l \times \Lambda_2)| \geq \delta |\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + 2^{-4} \alpha_1 \eta \beta_1 |\Lambda'| |\Lambda_1| |\Lambda_2|. \]  

This implies that, there exists a number \( l \in B'^c \setminus B_1 \) such that
\[ |A \cap (\lambda_l \times \Lambda_2)| > (\delta + 2^{-5} \alpha_1 \eta)|\lambda_l \cap E_1| |\Lambda_2 \cap E_2|. \]  

Put \( \tilde{\Lambda} = \Lambda' \), \( y_1 = l_0 \) and \( F_1 = (\tilde{\Lambda} + l_0) \cap E_1 \). Since \( l_0 \notin B_1 \), it follows that \(|F_1| \geq \beta_1 |\tilde{\Lambda}|/2\). The set \( E_2 \) is \((\alpha_0, 2^{-10} \varepsilon^2)\)-uniform. This yields that there exists a number \( a \) such that \( F_2 = (\tilde{\Lambda} + a) \cap E_2 \) has the cardinality at least \( \beta_2 |\tilde{\Lambda}|/2 \) and for \( \tilde{y} = (l_0, a) \) we have
\[ |A \cap (\tilde{\Lambda} + \tilde{y})| > (\delta + 2^{-6} \alpha_1 \eta) |F_1| |F_2|. \]

and the theorem is proven.

Let \( f(\tilde{x}) \) be the balanced function of \( A \). There exists \( l_0 \in B' \) such that
\[ |A \cap (\lambda_{l_0} \times \Lambda_2)| > (\delta - \eta)|\lambda_{l_0} \cap E_1| |\Lambda_2 \cap E_2|. \]

If
\[ |A \cap (\lambda_{l_0} \times \Lambda_2)| \geq (\delta + \eta)|\lambda_{l_0} \cap E_1| |\Lambda_2 \cap E_2|, \]  

(4.28)
then the theorem is proven.

Hence there exists $l_0 \in B'$ such that
\[ | \sum_{r,m} f(r,m)\Lambda_0(r)\Lambda_2(m) | < \eta|\Lambda_0 \cap E_1||\Lambda_2 \cap E_2| . \] (4.29)

Let $\Lambda_0 = \Lambda' + l_0$. Put $\nu_i = \Lambda'' + i$, $i \in \Lambda_0$ and $\mu_j = \Lambda'' + j$, $j \in \Lambda_2$. Consider the sum
\[ \sigma^* = \sum_{i \in \Lambda_0} \sum_{j \in \Lambda_2} \sum_k \sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) . \] (4.30)

Suppose that $i$ and $j$ are fixed in the sum (4.30). Using Lemma 2.3, we obtain that $k$ runs a set of cardinality at most $2|\Lambda_0|$. Besides that if $i$, $j$, $k$ are fixed, then $m$, $r$ run sets of size at most $|\Lambda''|$. Using Lemma 2.3 once again, we obtain
\[ \sigma^* = |\Lambda''|^2 \sum_k \sum_m \sum_{r \in \Lambda_0} f(r,m)\Lambda_0(m-k)\Lambda_2(k+r) + \theta\Lambda_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2| , \] (4.31)

where $|\theta| \leq 1$. Let $\Lambda_3 = \Lambda_2 - \Lambda' - l_0$. Using Lemma 2.3, we get $|\Lambda_2| \leq |\Lambda_3| \leq (1 + \alpha_0^2)|\Lambda_2|$. Note that $k$ belongs to the set $\Lambda_3$ in (4.31). If $k \in \Lambda_3 - l_0$, then $\Lambda_2(k+r) = 1$, for all $r \in \Lambda_0$. If $k$ is fixed in (4.31), then $r$ and $m$ run sets of cardinality at most $|\Lambda_0|$. It follows that
\[ \frac{\sigma^*}{|\Lambda''|^2} = \sum_{k \in (\Lambda_2 - l_0)} \sum_m \sum_{r \in \Lambda_0} f(r,m)\Lambda_0(m-k) + \sum_{k \in (\Lambda_2 - l_0)} \sum_m \sum_{r \in \Lambda_0} f(r,m)\Lambda_0(m-k)\Lambda_2(k+r) = \sum_{k \in (\Lambda_2 - l_0)} \sum_m \sum_{r \in \Lambda_0} f(r,m)\Lambda_0(m-k) + \alpha_0^2\theta_1|\Lambda_0|^2|\Lambda_2| = \sum_{k \in (\Lambda_2 - l_0)} \sum_m \sum_{r \in \Lambda_0} f(r,m)\Lambda_0(m-k) + 2\alpha_0^2\theta_2|\Lambda_0|^2|\Lambda_2| , \]

where $|\theta_1|, |\theta_2| \leq 1$. Using (4.29) we get
\[ |\sigma^*| < \eta|\Lambda''|^2|\Lambda_0||\Lambda_0 \cap E_1||\Lambda_2 \cap E_2| + 4\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2| \] (4.32)

If $j$ is fixed, then $k$ runs a set $-\Lambda_0 + j + \Lambda''$ in (4.30). Clearly, the cardinality of this set does not exceed $(1 + \alpha_0^2)|\Lambda'|$. Hence, replacing $4\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|$ in (4.32) by $8\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|$, we can assume that $k$ runs $-\Lambda_0 + j$ in (4.30).

Since $l \in B'$, it follows that $\beta_1|\Lambda_0|/2 \leq |\Lambda_0 \cap E_1| \leq 2\beta_1|\Lambda_0|$. Besides that $16\alpha_0^2 < \eta\beta_1\beta_2$. This implies that
\[ | \sum_{i \in \Lambda_0} \sum_{j \in \Lambda_2} \sum_{k \in -\Lambda_0 + j} \sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) | < 2\eta|\Lambda''|^2|\Lambda_0| \cdot |\Lambda_0 \cap E_1| \cdot |\Lambda_2 \cap E_2| \leq 4\eta\beta_1\beta_2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2| . \] (4.33)
Let
\[ \Omega = \{ j \in \Lambda_2 \mid \frac{1}{|\Lambda'|} \sum_{k \in \Lambda' + j} |\delta_{\Lambda'' + k}(E_2) - \beta_2|^2 \geq 4\alpha_0^{1/2} \} \],
and \( G = \Lambda_2 \setminus \Omega \).

Since \( E_2 \) is \((\alpha_0, \varepsilon')\)-uniform, it follows that \( |\Omega| \leq 8\alpha_0^{1/2} |\Lambda_2| \). Let \( i \in \Lambda_0 \) be fixed. Let
\[ \Omega(i) = \{ j \in \Lambda_2 \mid \frac{1}{|\Lambda'|} \sum_{k \in \Lambda_0 + j} |\delta_{\Lambda'' + i + k}(E_2) - \beta_2|^2 \geq 4\alpha_0^{1/2} \} \],
and \( G(i) = \Lambda_2 \setminus \Omega(i) \).

Since \( i \) belongs to \( \Lambda_0 \), this implies that a number \( a = l_0 - i \) belongs to \( \Lambda' \). Using Lemma 2.3 for \( \Lambda_2 \) and its \( \varepsilon \)-attendant \( \Lambda' \), we get \((G \cap \Lambda_2^-) + a \subseteq \Lambda_2 \) and
\[ |\Lambda_2 \cap (G + a)| \geq |\Lambda_2 \cap ((G \cap \Lambda_2^-) + a)| = |(G \cap \Lambda_2^-) + a| = |G \cap \Lambda_2^-| \geq |G| - 8\alpha_0^{1/2} |\Lambda_2| .
\]

Hence \( |\Omega(i)| \leq 8\alpha_0^{1/2} |\Lambda_2| \).

Since \( l_0 \in B' \), it follows that
\[ \frac{1}{|\Lambda'|} \sum_{k \in \Lambda'} |\delta_{\Lambda'' + k}(E_1 - l_0 \cap \Lambda') - \beta_1|^2 \leq 2^6 \alpha_0^{1/2} \tag{4.34} \]

It is clear that for any \( j \) the sum \( \text{(4.34)} \) equals
\[ \frac{1}{|\Lambda'|} \sum_{k \in -\Lambda_0 + j} |\delta_{\Lambda'' + j - k}(E_1 \cap \Lambda_0) - \beta_1|^2 . \]

Indeed
\[ \sum_{k \in -\Lambda_0 + j} |\delta_{\Lambda'' + j - k}(E_1 \cap \Lambda_0) - \beta_1|^2 = \sum_{k \in \Lambda' + l_0} |\delta_{\Lambda'' + k}(E_1 \cap \Lambda' + l_0) - \beta_1|^2 = \sum_{k \in \Lambda'} |\delta_{\Lambda'' + k}(E_1 - l_0 \cap \Lambda') - \beta_1|^2 \]

Let
\[ \Omega_1(i, j) = \{ k \in -\Lambda_0 + j : |\delta_{\Lambda'' + i + k}(E_2) - \beta_2| \geq 4\alpha_0^{1/8} \} , \]
\[ \Omega_2(i, j) = \{ k \in -\Lambda_0 + j : |\delta_{\Lambda'' + j - k}(E_1 \cap \Lambda_0) - \beta_1| \geq 4\alpha_0^{1/8} \} , \]

and
\[ \Omega_3(i, j) = \Omega_1(i, j) \cup \Omega_2(i, j) . \]

For all \( j \not\in \Omega(i) \) we have \( |\Omega_1(i, j)| \leq 2\alpha_0^{1/4} |\Lambda'| \). The inequality \( \text{(4.34)} \) implies that
\[ |\Omega_2(i, j)| \leq 4\alpha_0^{1/4} |\Lambda'| . \]

Hence \( |\Omega_3(i, j)| \leq 8\alpha_0^{1/4} |\Lambda'| \) if \( j \not\in \Omega(i) \).

Since \( l_0 \in B' \), it follows that
\[ \sigma = \sum_{i \in \Lambda_0} \sum_{j \in \Lambda_2} \sum_{k \in \Lambda_0} \mu_i(m - k)\nu_i(u - k) \left| \sum_r \mu_j(k + r)\tilde{f}_{i_0}(r, m)\tilde{f}_{i_0}(r, u) \right|^2 \geq \alpha \beta_1^2 \beta_2^2 |\Lambda''|^4 |\Lambda_0|^2 |\Lambda_2| , \tag{4.35} \]
where \( \tilde{f}_0 \) is a restriction of \( f \) to \( \Lambda_0 \times \Lambda_2 \). If \( j \) is fixed, then \( k \) runs \( -\Lambda_0 + j + \Lambda'' \) in (4.35). Clearly, the cardinality of this set does not exceed \((1 + \alpha_0^2)|\Lambda'|\). Hence, replacing \( \alpha \) by \( \alpha/2 \) in (4.35), we can assume that \( k \) runs \(-\Lambda_0 + j\) in (4.35). Using \(|\Omega(i)| \leq 8\alpha_0^{1/2}|\Lambda_2|\), we get

\[
\sigma = \sum_{i \in \Lambda_0} \sum_{j \notin \Omega(i)} \sum_{k} \sum_{m, u} \nu_i(m - k) \nu_i(u - k) \left| \sum_r \mu_j(k + r) \tilde{f}_0(r, m) \tilde{f}_0(r, u) \right|^2 \geq \frac{\alpha}{4} \beta_1^2 \beta_2^2 |\Lambda''|^4 |\Lambda_0|^2 |\Lambda_2|.
\]

(4.36)

Now we can prove the theorem.

Let \( J = \{ (i, j, k) | i \in \Lambda_0, j \notin \Omega(i), k \notin \Omega_0(i, j) \} \) such that

\[
\sum_{m, u} \nu_i(m - k) \nu_i(u - k) \left| \sum_r \mu_j(k + r) \tilde{f}_0(r, m) \tilde{f}_0(r, u) \right|^2 \geq \frac{\alpha}{64} \beta_1^2 \beta_2^2 |\Lambda''|^4 |\Lambda_0|^2 |\Lambda_2|.
\]

Using (4.36), we get

\[
\sum_{i \in \Lambda_0} \sum_{j \notin \Omega(i)} \sum_{k} \sum_{m, u} \nu_i(m - k) \nu_i(u - k) \left| \sum_r \mu_j(k + r) \tilde{f}_0(r, m) \tilde{f}_0(r, u) \right|^2 \geq \frac{\alpha}{8} \beta_1^2 \beta_2^2 |\Lambda''|^4 |\Lambda_0|^2 |\Lambda_2|.
\]

(4.37)

It follows that

\[
\sum_{(i, j, k) \in J} \sum_{m, u} \nu_i(m - k) \nu_i(u - k) \left| \sum_r \mu_j(k + r) \tilde{f}_0(r, m) \tilde{f}_0(r, u) \right|^2 \geq \frac{\alpha}{16} \beta_1^2 \beta_2^2 |\Lambda''|^4 |\Lambda_0|^2 |\Lambda_2|.
\]

(4.38)

Let us estimate the cardinality of \( J \). For any triple \( (i, j, k) \) belongs to \( J \) we have \( |E_2 \cap (\nu_i + k) - \beta_2| \Lambda''| \leq 4\alpha_0^{1/8} |\Lambda''| \) and \( |(E_1 \cap \Lambda_0) \cap (\nu_j - k) - \beta_1| \Lambda''| \leq 4\alpha_0^{1/8} |\Lambda''| \). Using (4.35), we get

\[
32|J| \cdot |\Lambda''|^4 \beta_1^2 \beta_2^2 \geq \frac{\alpha}{16} \beta_1^2 \beta_2^2 |\Lambda''|^4 |\Lambda_0|^2 |\Lambda_2|.
\]

(4.39)

This yields that \( |J| \geq 2^{-12} \alpha_0 |\Lambda_0|^2 |\Lambda_2| \).

Let us assume that for all \((i, j, k) \in J\) we have

\[
\sum_{m} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r) < -2^{15/\alpha} \beta_1 \beta_2 |\Lambda''|^2.
\]

(4.40)

Using (4.35), we get

\[
\sum_{(i, j, k) \in J} \sum_{m} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r) \geq 4\eta \beta_1 \beta_2 |\Lambda''|^2 |\Lambda_0|^2 |\Lambda_2|,
\]

(4.41)

where \( J = \{ (i, j, k) : (i, j, k) \in (\Lambda_0 \times \Lambda_2 \times (-\Lambda_0 + j) \setminus J \}. \) Since \(|\Omega(i)| \leq 8\alpha_0^{1/2} |\Lambda_2|, i \in \Lambda_0, \) it follows that

\[
\sum_{(i, j, k) \in J} \sum_{m} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r) \geq 2\eta \beta_1 \beta_2 |\Lambda''|^2 |\Lambda_0|^2 |\Lambda_2|.
\]

(4.42)
Hence, there exist \(i, j \notin \Omega(i)\) such that
\[
\sum_{k \in Q(i, j)} \sum_{m \in \Lambda_0} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r) \geq \frac{\eta}{2} |\beta_1 \beta_2| |\Lambda'|^2 |\Lambda_0|, \tag{4.43}
\]
where \(Q(i, j)\) is a subset of \(-\Lambda_0 + j\). Since \(j \notin \Omega(i)\), it follows that \(|\Omega_3(i, j)| \leq 8\alpha_{0}^{3/4} |\Lambda'|\). Hence
\[
\sum_{k \in Q(i, j) \setminus \Omega_3(i, j)} \sum_{m \in \Lambda_0} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r) \geq \frac{\eta}{4} |\beta_1 \beta_2| |\Lambda'|^2 |\Lambda_0|. \tag{4.44}
\]
This implies that there exists \(k \notin \Omega_3(i, j)\) such that
\[
\sum_{m \in \Lambda_0} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r) \geq \frac{\eta}{8} |\beta_1 \beta_2| |\Lambda''|^2. \tag{4.45}
\]
Put \(\tilde{\Lambda} = \Lambda''\), \(\tilde{\nu} = (j - k, k + i)\) and \(F_1 = (\tilde{\Lambda} + y_1) \cap (E_1 \cap \Lambda_0), F_2 = (\tilde{\Lambda} + y_2) \cap E_2\). Since \(k \notin \Omega_3(i, j)\), it follows that \(\beta_1|\Lambda''|/2 \leq |F_1| \leq 2\beta_1|\Lambda''|, \beta_2|\Lambda''|/2 \leq |F_2| \leq 2\beta_2|\Lambda''|\). Using this and (4.45), we get
\[
|A \cap (F_1 \times F_2)| = |A \cap ((\mu_j - k) \cap \Lambda_0) \cap ((\nu_i + k) \cap \Lambda_0)| \geq
\]
\[
\geq \delta |(\mu_j - k) \cap \Lambda_0| |(\nu_i + k) \cap \Lambda_0| + \frac{\eta}{8} |\beta_1 \beta_2| |\Lambda''|^2 \geq
\]
\[
\geq (\delta + \frac{\eta}{32}) |F_1| |F_2|. \tag{4.46}
\]
Hence, if for all \((i, j, k) \in J\) we have (4.40), then the theorem is proven.

Now assume that there exists a triple \((i, j, k) \in J\) such that
\[
\sum_{m \in \Lambda_0} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r) \geq -2^{15} \frac{\eta}{\alpha} |\beta_1 \beta_2| |\Lambda''|^2. \tag{4.47}
\]
We can assume that for all \((i, j, k) \in J\) we have
\[
|\sum_{m \in \Lambda_0} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r)| \leq 2^{15} \frac{\eta}{\alpha} |\beta_1 \beta_2| |\Lambda''|^2. \tag{4.48}
\]
Indeed, if
\[
\sum_{m \in \Lambda_0} \sum_{r \in \Lambda_0} f(r, m) \nu_i(m - k) \mu_j(k + r) > 2^{15} \frac{\eta}{\alpha} |\beta_1 \beta_2| |\Lambda''|^2,
\]
then we might apply the same reasoning as above. For sets \(\tilde{\Lambda}_1 = \Lambda''\), \(\tilde{\Lambda}_2 = \Lambda''\), a vector \(\tilde{\nu} = (j - k, k + i)\) and \(F_1 = (\tilde{\Lambda}_1 + y_1) \cap (E_1 \cap \Lambda_0), F_2 = (\tilde{\Lambda}_2 + y_2) \cap E_2\) we have \(|F_1| \geq \beta_1|\tilde{\Lambda}_1|/2, |F_2| \geq \beta_2|\tilde{\Lambda}_2|/2\) and
\[
|A \cap (F_1 \times F_2)| \geq (\delta + \frac{2^{6} \eta}{\alpha^{2}}) |F_1| |F_2|. \tag{4.49}
\]
Since \((i, j, k) \in J\), it follows that
\[
\sum_{m, u \in \nu_i + k} \sum_{r \in \mu_j - k} f_{\tilde{l}_0}(r, m) f_{\tilde{l}_0}(r, u) \geq 2^{-6} \alpha^{-2} |\beta_1 \beta_2|^4 |\Lambda''|^4. \tag{4.48}
\]
Note that \(m, u\) belong to \(\nu_i + k \cap \Lambda_0\) in (4.48) and \(r\) belongs to a set \(\mu_j - k \cap \Lambda_0\). Put \(L_1 = \mu_j - k \cap \Lambda_0, L_2 = \nu_i + k \cap \Lambda_2, E'_1 = E_1 \cap L_1\) and \(E'_2 = E_2 \cap L_2\). We can assume that \(f_{\tilde{l}_0}\) is zero outside \(L_1 \times L_2\) in (4.48). Let \(A_1 = A \cap (L_1 \times L_2), \delta_1 = \delta_{E'_1 \times E'_2}(A), \delta_2 = \delta_{E_1 \cap E_2}(A).

and \( f_1 \) be a balanced function of \( A_1 \). Using \((4.47)\), we get \( |\delta_1 - \delta| \leq 2^{20 \frac{n}{\alpha}} \). We have \( k \notin \Omega_3(i, j) \). Using this, we obtain

\[
\| \hat{f}_{i_0} - f_1 \|^2 = |E'_1|^2 |E'_2|^2 (\delta_1 - \delta)^2 \leq 2^{44} \beta_1^3 \beta_2^3 \gamma^2 |\Lambda''|^2. \tag{4.49}
\]

Using Lemma \((3.2)\) we get

\[
\sum_{m, u \in \nu_i + k} \sum_{r \in \mu_j - k} f_1(r, m) f_1(r, u) \geq 2^{-7} \alpha \beta_1^2 \beta_2^2 |\Lambda''|^4. \tag{4.50}
\]

Since \( k \notin \Omega_3(i, j) \), it follows that \( 2^{-1} \beta_1 |\Lambda''| \leq |E'_1| \leq 2 \beta_1 |\Lambda''|, \) \( 2^{-1} \beta_2 |\Lambda''| \leq |E'_2| \leq 2 \beta_2 |\Lambda''| \). Hence

\[
\sum_{m, u \in \nu_i + k} \sum_{r \in \mu_j - k} f_1(r, m) f_1(r, u) \geq 2^{-11} \alpha |E'_1|^2 |E'_2|^2. \tag{4.51}
\]

Using Proposition \((4.1)\) we obtain sets \( F_1 \subseteq E'_1 \subseteq \mu_j - k, F_2 \subseteq E'_2 \subseteq \nu_i + k \) such that

\[
|A \cap (F_1 \times F_2)| \geq |A_1 \cap (F_1 \times F_2)| \geq (\delta_1 + 2^{-37} \alpha^2 \delta^2) |F_1||F_2| \geq (\delta + 2^{-40} \alpha^2 \delta^2) |F_1||F_2|. \tag{4.52}
\]

and

\[
|F_1| \geq 2^{-40} \alpha^2 \delta^2 |E'_1| \geq 2^{-300} \delta^3 \beta_1 |\Lambda''|, \quad i = 1, 2.
\]

Put \( \hat{\Lambda} = \Lambda'' \), \( \hat{y} = (j - k, k + i) \) and \( F_1 = (\hat{\Lambda}_1 + y_1) \cap (E_1 \cap A_0), F_2 = (\hat{\Lambda}_2 + y_2) \cap E_2 \). The sets \( \hat{\Lambda} \) and \( F_1, F_2 \) satisfy \((4.20), (4.21)\). This concludes the proof.

5. On dense subsets of Bohr sets.

The following lemmas were proven in \([27]\).

**Lemma 5.1.** Let \( \Lambda \) be a Bohr set, \( \Lambda' \) be an \( \varepsilon \)-attendant of \( \Lambda, \varepsilon = \kappa/(100d) \), and \( Q \) be a subset of \( \Lambda \). Let \( g : 2^G \times (G \times G) \to \mathbb{D} \) be the function such that \( g(\Lambda, \bar{x}) = \delta_{\Lambda + \bar{x}}^2(Q) \). Then

\[
\frac{1}{|\Lambda|^2} \sum_{\bar{x} \in \Lambda} g(\Lambda', \bar{x}) \geq g(\Lambda, 0) - 8\kappa. \tag{5.1}
\]

**Lemma 5.2.** Let \( \Lambda \) be a Bohr set, \( \Lambda' \) be an \( \varepsilon \)-attendant of \( \Lambda, \varepsilon = \kappa/(100d) \), \( \alpha > 0 \) be a real number, and \( Q \) be a subset of \( \Lambda \), \( |Q| = \delta |\Lambda| \). Suppose that

\[
\frac{1}{|\Lambda|^2} \sum_{\bar{n} \in \Lambda} |\delta_{\Lambda' + \bar{n}}^2(Q) - \delta|^2 \geq \alpha. \tag{5.2}
\]

Then

\[
\sum_{\bar{n} \in \Lambda} \delta_{\Lambda' + \bar{n}}^2(Q) \geq \delta^2 + \alpha - 4\kappa. \tag{5.3}
\]
NOTE. Clearly, the one–dimension analogs of Lemma 5.1 and Lemma 5.2 take place.

Also, in [27] was proven a corollary.

**Corollary 5.1.** Let $\Lambda$ be a Bohr set, $\alpha > 0$ be a real number, and $E_1, E_2$ be sets, $|E_1 \cap \Lambda| = \beta_1|\Lambda|$, $|E_2 \cap \Lambda| = \beta_2|\Lambda|$. Suppose that either $E_1$ or $E_2$ does not satisfy (2.13). Let $\Lambda'$ be an arbitrary $(2^{-10}\alpha^2\beta_1^2\beta_2^2)/(100d)$–attendant set of $\Lambda$. Then

$$\frac{1}{|\Lambda|^2} \sum_{n \in \Lambda} \delta_{\Lambda + n}^{\beta_1\beta_2}(E_1 \times E_2) \geq \beta_1^2\beta_2^2(1 + \frac{\alpha^2}{2}). \quad (5.4)$$

The following lemma was proven by J. Bourgain in [3]. We give his proof for the sake of completeness.

**Lemma 5.3.** Let $\Lambda = \Lambda(S, \varepsilon)$ be a Bohr set, $|S| = d \in \mathbb{N}$, $\alpha > 0$ be a real number, and $Q$ be a set, $|Q \cap \Lambda| = \delta|\Lambda|$. Suppose that

$$||(Q \cap \Lambda - \delta \Lambda)\|_\infty \geq \alpha|\Lambda|. \quad (5.5)$$

Then there exists a Bohr set $\Lambda' = \Lambda(S', \varepsilon')$, $|S'| = d + 1$ such that $\Lambda'$ is an $\varepsilon_1$–attendant of $\Lambda$, $\varepsilon_1 = \frac{\alpha}{100d}$, $\kappa \leq \alpha/32$ and

$$\frac{1}{|\Lambda|} \sum_{n \in \Lambda} |\delta_{\Lambda' + n}(Q) - \delta|^2 \geq \frac{\alpha^2}{4}. \quad (5.6)$$

**Proof.** Let $Q_1 = Q \cap \Lambda$. Using (5.5), we obtain

$$|\tilde{Q}_1(\xi_0) - \delta \Lambda(\xi_0)| \geq \alpha|\Lambda|, \quad (5.7)$$

where $\xi_0 \in \tilde{G}$. We have $\Lambda = \Lambda_{S, \varepsilon}$, where $S \subseteq \tilde{G}$. Put $S' = S \cup \{\xi_0\} \subset \tilde{G}$ and

$$\Lambda' = \Lambda_{S', \varepsilon'}$$

be an $\varepsilon_1$–attendant of $\Lambda$. Using Lemma 2.3 we get

$$\tilde{Q}_1(\xi_0) = \sum_n Q(n)\Lambda(n)e^{-2\pi i (\xi_0 \cdot n)} = \frac{1}{|\Lambda'|} \sum_n (\Lambda * \Lambda')(n)Q(n)e^{-2\pi i (\xi_0 \cdot n) + 2\kappa \vartheta|\Lambda|},$$

where $|\vartheta| \leq 1$. We have

$$\tilde{Q}_1(\xi_0) = \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n - m)\Lambda(m)Q(n)e^{-2\pi i (\xi_0 \cdot n)} + 2\kappa \vartheta|\Lambda| =$$

$$= \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n - m)\Lambda(m)Q(n)e^{-2\pi i (\xi_0 \cdot m)} +$$

$$+ \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n - m)\Lambda(m)Q(n)[e^{-2\pi i (\xi_0 \cdot n)} - e^{-2\pi i (\xi_0 \cdot m)}] + 2\kappa \vartheta|\Lambda| =$$

$$= \sum_{m \in \Lambda} \delta_{\Lambda' + m}(Q)e^{-2\pi i (\xi_0 \cdot m)} + \vartheta \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n - m)\Lambda(m)Q(n)[e^{-2\pi i (\xi_0 \cdot (n - m))} - 1] +$$

...
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+ 2\kappa \vartheta |\Lambda| = \sum_{m \in \Lambda} \delta_{\Lambda}^{(m)}(Q)e^{-2\pi i (\xi_0 m)} + (14\kappa \vartheta_1 + 2\kappa \vartheta)|\Lambda|, \quad (5.8)

where |\vartheta_1| \leq 1. Using (5.5) and (5.8), we obtain

\[ \left| \sum_{m \in \Lambda} \delta_{\Lambda}^{(m)}(Q)e^{-2\pi i (\xi_0 m)} - \delta \sum_{m \in \Lambda} e^{-2\pi i (\xi_0 m)} \right| \geq \frac{\alpha}{2} |\Lambda|. \quad (5.9) \]

Hence

\[ \sum_{m \in \Lambda} |\delta_{\Lambda}^{(m)}(Q) - \delta| \geq \frac{\alpha}{2} |\Lambda|. \quad (5.10) \]

Using the Cauchy–Schwartz inequality, we get

\[ \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda} |\delta_{\Lambda^{\vec{n}}}(Q) - \delta|^2 \geq \frac{\alpha^2}{4}. \quad (5.11) \]

This completes the proof.

**Corollary 5.2.** Let \( \Lambda = \Lambda(S, \varepsilon) \) be a Bohr set, \( \alpha > 0 \) be a real number, and \( E_1, E_2 \) be sets, \( |E_1 \cap \Lambda| = \beta_1|\Lambda|, \ |E_2 \cap \Lambda| = \beta_2|\Lambda| \). Suppose that either \( E_1 \) or \( E_2 \) satisfies (5.13). Then there exists \( (2^{-10}\alpha^2\beta_1^2\beta_2^2)/(100d) \)–attendant set \( \Lambda' = \Lambda(S', \varepsilon') \) of the Bohr set \( \Lambda \) such that

\[ \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda} \delta_{\Lambda^{\vec{n}}}(E_1 \times E_2) \geq \beta_1^2 \beta_2^2 (1 + \frac{\alpha^2}{8}) \quad (5.12) \]

and

\[ |S'| = d + 1. \quad (5.13) \]

**Proof.** Let \( \vec{n} = (x, y) \), and \( \kappa = 2^{-10}\alpha^2\beta_1^2\beta_2^2 \). We have

\[ \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda} \delta_{\Lambda^{\vec{n}}}(E_1 \times E_2) = \left( \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \delta_{\Lambda}^{(x)}(E_1) \right) \left( \frac{1}{|\Lambda|} \sum_{y \in \Lambda} \delta_{\Lambda}^{(y)}(E_2) \right) \quad (5.14) \]

We can assume without loss of generality that \( E_1 \) satisfies (5.5). Using Lemma 5.3 and Lemma 5.2, we obtain

\[ \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \delta_{\Lambda}^{(x)}(E_1) \geq \beta_1^2 + \frac{\alpha^2}{4} - 4\kappa. \quad (5.15) \]

Let us estimate the second term in (5.14). Using Lemma 5.1, we get

\[ \frac{1}{|\Lambda|} \sum_{y \in \Lambda} \delta_{\Lambda}^{(y)}(E_2) \geq \beta_2^2 - 8\kappa. \quad (5.16) \]

Combining (5.15) and (5.16), we obtain

\[ \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda} \delta_{\Lambda^{\vec{n}}}(E_1 \times E_2) \geq (\beta_1^2 + \frac{\alpha^2}{4} - 4\kappa)\beta_2^2 - 8\kappa) \geq \beta_1^2 \beta_2^2 (1 + \frac{\alpha^2}{8}). \]

This concludes the proof.

We shall say that the set \( S' \) from (5.13) is constructed by Corollary 5.2.

Clearly, all lemmas of this section apply to translations of Bohr sets.
Our further arguments and arguments from [27] are particularly the same.

Let $\Lambda$ be a union of a family of Bohr sets $\Lambda_0, \Lambda_1(\vec{x}_0), \ldots, \Lambda_n(\vec{x}_0, \ldots, \vec{x}_{n-1})$ and a sequence of some translations of Bohr sets $\Lambda_0, \Lambda_1(\vec{x}_0), \ldots, \Lambda_n(\vec{x}_0, \ldots, \vec{x}_{n-1})$ such that

$\Lambda_1(\vec{x}_0)$ and $\Lambda_1^*(\vec{x}_0)$ are defined iff $\vec{x}_0 \in \Lambda_0$

$\Lambda_2(\vec{x}_0, \vec{x}_1)$ and $\Lambda_2^*(\vec{x}_0, \vec{x}_1)$ are defined iff $\vec{x}_1 \in \Lambda_1(\vec{x}_0), \vec{x}_0 \in \Lambda_0$

$\ldots$

$\Lambda_n(\vec{x}_0, \ldots, \vec{x}_{n-1})$ and $\Lambda_n^*(\vec{x}_0, \ldots, \vec{x}_{n-1})$ are defined iff

$\vec{x}_{n-1} \in \Lambda_{n-1}(\vec{x}_0, \ldots, \vec{x}_{n-2}), \vec{x}_{n-2} \in \Lambda_{n-2}(\vec{x}_0, \ldots, \vec{x}_{n-3}), \ldots, \vec{x}_0 \in \Lambda_0.$

(5.17)

Let $m \geq 0$ be an integer number and $\Lambda$ be a family of Bohr sets satisfies (5.17).

Let $g : 2^G \times (G \times G) \to \mathcal{D}$ be a function. Let us define the index of $g$, respect $\Lambda$, for all $k = 0, \ldots, m$ by

$$\text{ind}_k(\Lambda)(g) = \frac{1}{|\Lambda_0|^2} \sum_{\vec{x}_0 \in \Lambda_0} \frac{1}{|\Lambda_1(\vec{x}_0)|^2} \sum_{\vec{x}_1 \in \Lambda_1(\vec{x}_0)} \ldots \frac{1}{|\Lambda_k(\vec{x}_0, \ldots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0, \ldots, \vec{x}_{k-1})} g(\Lambda_k^*(\vec{x}_0, \ldots, \vec{x}_{k-1}), \vec{y}).$$

(5.18)

Let $M_k = M_k(\vec{x}_0, \ldots, \vec{x}_{k-1})$ be the family of sets such that $M_k(\vec{x}_0, \ldots, \vec{x}_{k-1}) \subseteq \Lambda_k(\vec{x}_0, \ldots, \vec{x}_{k-1})$ for all $(\vec{x}_0, \ldots, \vec{x}_{k-1}).$ For any $k = 0, \ldots, m$ by $\text{ind}_k(\Lambda, M)(g)$ define the following expression

$$\text{ind}_k(\Lambda, M)(g) = \frac{1}{|\Lambda_0|^2} \sum_{\vec{x}_0 \in \Lambda_0} \frac{1}{|\Lambda_1(\vec{x}_0)|^2} \sum_{\vec{x}_1 \in \Lambda_1(\vec{x}_0)} \ldots \frac{1}{|\Lambda_k(\vec{x}_0, \ldots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in M_k(\vec{x}_0, \ldots, \vec{x}_{k-1})} g(\Lambda_k^*(\vec{x}_0, \ldots, \vec{x}_{k-1}), \vec{y}).$$

(5.19)

Clearly, we have $|\text{ind}_k(\Lambda, M)(g)| \leq 1$, for any natural $k \geq 0$, a family $M_k$ and a function $g : 2^G \times (G \times G) \to \mathcal{D}$.

The following simple lemma was proven in [27].

**Lemma 5.4.** Let $Q$ be a subset of $\Lambda_0 \times \Lambda_0$, and $|Q| = \delta|\Lambda_0|^2$. Suppose that $\Lambda_k^*(\vec{x}_0, \ldots, \vec{x}_{k-1})$ is an arbitrary $\varepsilon$-attendant of $\Lambda_k(\vec{x}_0, \ldots, \vec{x}_{k-1})$, $\varepsilon = \kappa/(100d)$. Let $g(M, \vec{x}) = \delta_M + \bar{\varepsilon}(Q)$. Then for all $k = 0, \ldots, n$ we have

$$|\text{ind}_k(\Lambda)(g) - \delta| \leq 4\kappa(k + 1).$$

(5.20)

The next result is the main in this section.

**Proposition 5.1.** Let $\Lambda = \Lambda(S, \varepsilon_0)$ be a Bohr set, $|S| = d$, and $\vec{s} = (s_1, s_2)$ be a vector. Let $\varepsilon, \sigma, \tau, \delta \in (0, 1)$ be real numbers, $E_1, E_2$ be sets, $E_i = \beta_i|\Lambda|$, $i = 1, 2$. 

Suppose that \( E = E_1 \times E_2 \) is a subset of \((\Lambda + s_1) \times (\Lambda + s_2), A \subseteq E, \delta(E) = \delta + \tau, \) and \( \varepsilon \leq \kappa/(100d), \) \( \kappa = 2^{-100}(\tau \beta \gamma)\beta^2. \) Let
\[
N \geq (2^{-100} \varepsilon \delta)^{-2^{100}(\tau \beta \gamma)\beta^{-3} + d}.
\]
and \( \sigma \leq 2^{-100}\tau \beta \gamma. \) Then there exists a Bohr set \( \Lambda = \Lambda(S', \varepsilon'), |S'| = D, \)
\( D \leq 2^{30}(\tau \beta \gamma)\beta^{-3} + d, \varepsilon' \leq (2^{-100} \varepsilon)^D \varepsilon_0, \) and a vector \( \vec{t} = (t_1, t_2) \) such that if \( E_1' = (E_1 - t_1) \cap \Lambda', E_2' = (E_2 - t_2) \cap \Lambda', E' = E_1' \times E_2', \) then
1) \( |E'| \geq \beta_1 \beta_2 |\Lambda'|/16; \)
2) \( E_1', E_2' \) are \((\sigma, \varepsilon)-\text{uniform subsets of } \Lambda'; \)
3) \( \delta(E) / \beta - \delta + \tau/16. \)

**Proof.** Let \( \beta = \beta_1 \beta_2, \) and \( \tilde{E}_1 = E_1 - s_1, \tilde{E}_2 = E_2 - s_2, \tilde{E} = \tilde{E}_1 \times \tilde{E}_2. \) If the sets \( \tilde{E}_1, \tilde{E}_2 \) are \((\sigma, \varepsilon)-\text{uniform subsets of } \Lambda, \) then Proposition 5.1 is proven.

Suppose that \( \tilde{E}_1, \tilde{E}_2 \) are not \((\sigma, \varepsilon)-\text{uniform subsets of } \Lambda. \) We shall construct a family of Bohr sets \( \Lambda \) such that \( \Lambda \) satisfies the conditions 5.17. The proof of Proposition 5.1 is a sort of an algorithm. At the first step of our algorithm we put \( \Lambda_0 = \Lambda = \Lambda(S, \varepsilon_0). \) If either \( \tilde{E}_1 \) or \( \tilde{E}_2 \) does not satisfy 5.17 with \( \alpha = \sigma/2, \) then let \( \Lambda_0^\ast \) be an \( \varepsilon \)-attendant of \( \Lambda_0 \) such that \( \Lambda_0^\ast \) is constructed by Corollary 5.2. In the other cases let \( \Lambda_0^\ast \) be an \( \varepsilon \)-attendant of \( \Lambda_0 \) with the same set \( S \) to be chosen later.
Define
\[
R_0 = \{ \vec{p} = (p_1, p_2) \in \Lambda_0 \mid \tilde{E}_1 - p_1, \tilde{E}_2 - p_2 \text{ are } (\sigma, \varepsilon)-\text{uniform in } \Lambda_0^\ast \}
\]
and \( \overline{R}_0 = (\Lambda_0 \times \Lambda_0) \setminus R_0. \)

Let \( \tilde{\Lambda} \) be an arbitrary Bohr set, and \( \vec{n} \in G \times G \) be an arbitrary vector. Put \( g(\tilde{\Lambda}, \vec{n}) = \delta_{\tilde{\Lambda} + \vec{n}}^2(\tilde{E}), g_1(\tilde{\Lambda}, \vec{x}) = \delta_{\tilde{\Lambda} + \vec{x}}(A), g_2(\tilde{\Lambda}, \vec{n}) = \delta_{\tilde{E} \cap \tilde{\Lambda} + \vec{n}}(A) \) and \( g_3(\tilde{\Lambda}, \vec{n}) = \delta_{\tilde{E} \cap \tilde{\Lambda} + \vec{n}}((\tilde{E}). \) Clearly, \( g(\tilde{\Lambda}, \vec{n}) = g_3(\tilde{\Lambda}, \vec{n}) \) and \( g_1(\tilde{\Lambda}, \vec{x}) \leq g_3(\tilde{\Lambda}, \vec{n}). \) Besides that, we have
\[
g_1(\tilde{\Lambda}, \vec{n}) = g_2(\tilde{\Lambda}, \vec{n}) g_3(\tilde{\Lambda}, \vec{n}).
\]
Let \( \Lambda_0 = \{ \Lambda_0 \}. \) If \( \text{ind}_0(\Lambda_0, \overline{R}_0)(g_3) < \tau \beta/4, \) then we stop the algorithm at step 0.

Using Lemma 2.24 and the Cauchy–Schwartz inequality, we get
\[
\text{ind}_0(\Lambda_0)(g) \geq \left( \frac{1}{|\Lambda_0|^2} \sum_{\vec{y} \in \Lambda_0} \delta_{\Lambda_0^\ast + \vec{y}}(\tilde{E}) \right)^2 \geq \beta/2.
\]

Let after the \( k \)th step of the algorithm the family of Bohr sets \( \Lambda_k \) has been constructed, \( k \geq 0. \)

Let
\[
\Lambda_{k+1}(\vec{x}_0, \ldots, \vec{x}_k) = \Lambda_k^\ast(\vec{x}_0, \ldots, \vec{x}_{k-1}) + \vec{x}_k, \vec{x}_k \in \Lambda_k(\vec{x}_0, \ldots, \vec{x}_{k-1}).
\]
Let \( \vec{x}_k = (a, b), \) and \( \Lambda_k^\ast = \Lambda_k^\ast(\vec{x}_0, \ldots, \vec{x}_{k-1}). \) If either \( (\tilde{E}_1 - a) \cap \Lambda_k^\ast \) or \( (\tilde{E}_2 - b) \cap \Lambda_k^\ast \) does not satisfy 5.17 with \( \alpha = \sigma/2, \) then let \( \Lambda_{k+1}(\vec{x}_0, \ldots, \vec{x}_k) \) be an \( \varepsilon \)-attendant of \( \Lambda_k^\ast(\vec{x}_0, \ldots, \vec{x}_{k-1}) \) such that \( \Lambda_{k+1}(\vec{x}_0, \ldots, \vec{x}_k) \) is constructed by Corollary 5.2. In the other cases let \( \Lambda_k^\ast(\vec{x}_0, \ldots, \vec{x}_k) \) be an \( \varepsilon \)-attendant of \( \Lambda_k^\ast(\vec{x}_0, \ldots, \vec{x}_k) \) with the same generative vector.

By \( R_{k+1}(\vec{x}_0, \ldots, \vec{x}_k), \overline{R}_{k+1}(\vec{x}_0, \ldots, \vec{x}_k) \) denote the sets
\[
R_{k+1}(\vec{x}_0, \ldots, \vec{x}_k) = \{ \vec{p} = (p_1, p_2) \in \Lambda_k^\ast(\vec{x}_0, \ldots, \vec{x}_{k-1}) + \vec{x}_k \mid \tilde{E}_1 - p_1, \tilde{E}_2 - p_2
\]
Let

\[ \alpha \]

\[ \beta \]

Let us consider the following situation: either \((\tilde{\epsilon} \ast \cdot \cdot \cdot \ast) \) with \( \Lambda \)

is constructed by Corollary 5.2. Using Corollary 5.2, we get

\[ \text{Case 1: either } (\tilde{\epsilon} \ast \cdot \cdot \cdot \ast) \leq \text{Corollary 5.2}. \]

By \( E_k(x_0, \ldots, \tilde{x}_{k-1}) \) denote the sets

\[ E_k(x_0, \ldots, \tilde{x}_{k-1}) = \]

\[ \{ \bar{y} = (p_1, p_2) \in \Lambda_k^*(x_0, \ldots, \tilde{x}_{k-2}) + \tilde{x}_{k-1} \mid \delta_{\Lambda_k^*}(x_0, \ldots, \tilde{x}_{k-1}) + \bar{y} (\tilde{E}_1 \times \tilde{E}_2) < \beta / 16 \}. \]

Obviously, \( E_k(x_0, \ldots, \tilde{x}_{k-1}) \subseteq R_k(x_0, \ldots, \tilde{x}_{k-1}), k = 0, 1, \ldots \)

Let \( \Lambda'_{k+1} = \{ \Lambda_{k+1}(x_0, \ldots, \tilde{x}_{k}) \} \), \( \bar{x} \in \Lambda_k(x_0, \ldots, \tilde{x}_{k-1}) \), and \( \Lambda_{k+1} = \{ \Lambda_k, \Lambda'_{k+1} \} \).

Let \( \delta_{\Lambda_k^*} = \delta_{\Lambda_k^*}(x_0, \ldots, \tilde{x}_{k}) \), and \( \beta_k' = \delta_{\Lambda_k^*}(\tilde{E}_1), \beta_k'' = \delta_{\Lambda_k^*}(\tilde{E}_2) \).

Suppose \( \tilde{x}_{k-1} = (a', b') \) belongs to \( R_k(x_0, \ldots, \tilde{x}_{k-2}) \). Note that \( \tilde{x}_{k-1} \) does not belong to \( E_k(x_0, \ldots, \tilde{x}_{k-2}) \). Let us consider three cases.

Case 1: either \((\tilde{E}_1 - a') \cap \Lambda_k^* \) or \((\tilde{E}_2 - b') \cap \Lambda_k^* \) does not satisfy \( (3.12) \).

Case 2: either \((\tilde{E}_1 - a') \cap \Lambda_k^* \) or \((\tilde{E}_2 - b') \cap \Lambda_k^* \) does not satisfy \( (3.13) \).

Case 3: either \((\tilde{E}_1 - a') \cap \Lambda_k^* \) or \((\tilde{E}_2 - b') \cap \Lambda_k^* \) does not satisfy \( (3.14) \). Note that \( \alpha = \sigma \) in all these cases.

Let us consider the following situation: either \((\tilde{E}_1 - a') \cap \Lambda_k^* \) or \((\tilde{E}_2 - b') \cap \Lambda_k^* \) does not satisfy \( (3.14) \) with \( \alpha = 2^{-4} \sigma^{3/2} \). Let

\[ S_0 = \frac{1}{|\Lambda_k(x_0, \ldots, \tilde{x}_{k-1})|^2} \sum_{\bar{y} \in \Lambda_k(x_0, \ldots, \tilde{x}_{k-1})} g(\Lambda_k^*(x_0, \ldots, \tilde{x}_{k-1}), \bar{y}), \]

where \( \Lambda_k^*(x_0, \ldots, \tilde{x}_{k-1}) \) is an \( \epsilon \)-attendant of \( \Lambda_k(x_0, \ldots, \tilde{x}_{k-1}) \) such that \( \Lambda_k^*(x_0, \ldots, \tilde{x}_{k-1}) \)

is constructed by Corollary 5.2. Using Corollary 5.2 we get

\[ S_0 \geq g(\Lambda_k(x_0, \ldots, \tilde{x}_{k-1}), 0) (1 + 2^{-11} \sigma^3) = \]

\[ = g(\Lambda_k^*(x_0, \ldots, \tilde{x}_{k-2}), \tilde{x}_{k-1}) (1 + 2^{-11} \sigma^3). \]

Note that in this case, we have \( \dim \Lambda_k^*(x_0, \ldots, \tilde{x}_{k-1}) = \dim \Lambda_k(x_0, \ldots, \tilde{x}_{k-1}) + 1 \).

Suppose that either \((\tilde{E}_1 - a') \cap \Lambda_k^* \) or \((\tilde{E}_2 - b') \cap \Lambda_k^* \) does not satisfy \( (3.13) \) with \( \alpha = 2^{-4} \sigma^{3/2} \). Using Corollary 5.1 we obtain

\[ S_0 \geq g(\Lambda_k^*(x_0, \ldots, \tilde{x}_{k-2}), \tilde{x}_{k-1}) (1 + 2^{-11} \sigma^3). \]

In this case, we have \( \dim \Lambda_k^*(x_0, \ldots, \tilde{x}_{k-1}) = \dim \Lambda_k(x_0, \ldots, \tilde{x}_{k-1}) \).

Finally, suppose that either \((\tilde{E}_1 - a') \cap \Lambda_k^* \) or \((\tilde{E}_2 - b') \cap \Lambda_k^* \) does not satisfy \( (3.12) \) with \( \alpha = \sigma \). Note that \((\tilde{E}_1 - a') \cap \Lambda_k^* \) and \((\tilde{E}_2 - b') \cap \Lambda_k^* \)

satisfy \( (3.13) \) with \( \alpha = 2^{-4} \sigma^{3/2} \). Let \( \Lambda_k^* = \Lambda_k^*(x_0, \ldots, \tilde{x}_k) \). Define

\[ B_k(x_0, \ldots, \tilde{x}_{k-1}) = \{ \bar{y} = (p_1, p_2) \in \Lambda_k(x_0, \ldots, \tilde{x}_{k-1}) : \]

\[ \|((\tilde{E}_1 - p_1) - \beta_k' \Lambda_k^*)\|_{\infty} \geq \sigma|\Lambda_k| \text{ or } \|((\tilde{E}_2 - p_2) - \beta_k'' \Lambda_k^*)\|_{\infty} \geq \sigma|\Lambda_k^*| \}. \]

We have

\[ |B_k(x_0, \ldots, \tilde{x}_{k-1})| \geq \sigma|\Lambda_k(x_0, \ldots, \tilde{x}_{k-1})|^2. \]

Let

\[ \tilde{B}_k(x_0, \ldots, \tilde{x}_{k-1}) = \{ \bar{y} = (p_1, p_2) \in B_k(x_0, \ldots, \tilde{x}_{k-1}) : \]
ON A TWO–DIMENSIONAL ANALOG OF SZEMERÉDI’S THEOREM IN ABELIAN GROUPS

\[ |\delta_{\Lambda}^* (\vec{E}_1 - p_1) - \beta_k'| \leq \sigma/8 \quad \text{and} \quad |\delta_{\Lambda}^* (\vec{E}_2 - p_2) - \beta_k'| \leq \sigma/8 \].

For all \( \vec{p} \in \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \), we have either \((\vec{E}_1 - p_1) \cap \Lambda_k^* \neq \emptyset \) or \((\vec{E}_2 - p_2) \cap \Lambda_k^* \) does not \( \sigma/2 \)–uniform. The sets \((\vec{E}_1 - a') \cap \Lambda_{k-1}^* \) and \((\vec{E}_2 - b') \cap \Lambda_{k-1}^* \) satisfy (5.13) with \( \alpha = 2^{-4} \sigma^2/8 \). This implies that

\[ |\hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1})| \geq \frac{\sigma}{2} |\Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1})|^2. \]  

Suppose that

\[ g_3 (\Lambda_k^* (\vec{x}_0, \ldots, \vec{x}_{k-1})) = \beta_k' \beta_k'' \geq \tau \beta / 8. \]  

It follows from (5.28) that

\[ g_3 (\Lambda_k^* (\vec{x}_0, \ldots, \vec{x}_{k-1}), \vec{p}) \geq \beta_k' \beta_k'' - \sigma / 2 \geq \tau \beta / 16, \]  

for all \( \vec{p} \in \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \).

Let us consider the sum

\[ S = S (\vec{x}_0, \ldots, \vec{x}_{k-1}) = \sum_{\vec{y} \in \Lambda_{k+1} (\vec{x}_0, \ldots, \vec{x}_k)} g (\Lambda_k^* (\vec{x}_0, \ldots, \vec{x}_{k-1}), \vec{y}). \]

Write the sum \( S \) as \( S' + S'' \), where the summation in \( S' \) is taken over \( \vec{x}_k \in \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \) and the summation in \( S'' \) is taken over \( \vec{x}_k \in \Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \) \( \setminus \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \). Note that if \( \vec{x}_k \in \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \), then the Bohr set \( \Lambda_{k+1}^* (\vec{x}_0, \ldots, \vec{x}_k) \) is constructed by Corollary 5.2. Using this corollary, we obtain

\[ S' \geq \frac{1}{|\Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \setminus \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1})} g (\Lambda_k^* (\vec{x}_0, \ldots, \vec{x}_{k-1}), \vec{y}) (1 + \frac{\sigma^2}{32}). \]

Let us estimate the sum \( S'' \). Using Lemma 5.1, we get

\[ S'' \geq \frac{1}{|\Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \setminus \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1})} g (\Lambda_k^* (\vec{x}_0, \ldots, \vec{x}_{k-1}), \vec{y}) - 8 \kappa. \]

Combining (5.29), (5.30), (5.31) and (5.27), we have

\[ S \geq \frac{1}{|\Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \setminus \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1})} g (\Lambda_k^* (\vec{x}_0, \ldots, \vec{x}_{k-1}), \vec{y}) + \frac{1}{|\Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1})} 2^{-13} \tau^2 \beta^2 \sigma^2 - 2^4 \kappa \geq \frac{1}{|\Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k (\vec{x}_0, \ldots, \vec{x}_{k-1}) \setminus \hat{B}_k (\vec{x}_0, \ldots, \vec{x}_{k-1})} g (\Lambda_k^* (\vec{x}_0, \ldots, \vec{x}_{k-1}), \vec{y}) + 2^{-14} \tau^2 \beta^2 \sigma^3 - 2^4 \kappa \]

Using Lemma 5.1, we obtain

\[ S \geq g (\Lambda_{k-1}^* (\vec{x}_0, \ldots, \vec{x}_{k-2}), \vec{x}_{k-1}) + 2^{-14} \tau^2 \beta^2 \sigma^3 - 2^5 \kappa \geq \frac{1}{|\Lambda_k^* (\vec{x}_0, \ldots, \vec{x}_{k-2}), \vec{x}_{k-1}) + 2^{-15} \tau^2 \beta^2 \sigma^3 \geq g (\Lambda_{k-1}^* (\vec{x}_0, \ldots, \vec{x}_{k-2}), \vec{x}_{k-1}) (1 + 2^{-15} \tau^2 \beta^2 \sigma^3). \]
Thus if \( \mathbf{x}_{k-1} \) belongs to \( \overline{R}_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}) \) and \( \mathbf{x}_{k-1} \) satisfies (5.28), then we have

\[
S \geq g(\Lambda_{k-1}^*(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}), \mathbf{x}_{k-1})(1 + 2^{-15}r^2\beta^2\sigma^2) - 8\kappa. \tag{5.33}
\]

Now suppose that \( \mathbf{x}_{k-1} \) is an arbitrary vector, \( \mathbf{x}_{k-1} \in \Lambda_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}) \). Using Lemma [5.1] twice, we have

\[
S \geq g(\Lambda_{k-1}^*(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}), \mathbf{x}_{k-1}) - 16\kappa. \tag{5.34}
\]

Let us consider \( \text{ind}_{k+1}(\Lambda_{k+1})(g) \). We have

\[
\frac{1}{|A_0|^2} \sum_{\mathbf{x}_0 \in A_0} \frac{1}{|A_1(\mathbf{x}_0)|^2} \sum_{\mathbf{x}_1 \in A_1(\mathbf{x}_0)} \cdots \sum_{\mathbf{x}_{k-1} \in \Lambda_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2})} S(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}).
\]

By assumption \( \text{ind}_{k-1}(\Lambda_{k-1}, \overline{R}_{k-1})(g) \geq \tau\beta/4 \). In other words

\[
\frac{1}{|A_0|^2} \sum_{\mathbf{x}_0 \in A_0} \frac{1}{|A_1(\mathbf{x}_0)|^2} \sum_{\mathbf{x}_1 \in A_1(\mathbf{x}_0)} \cdots \sum_{\mathbf{x}_{k-1} \in \Lambda_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2})} g_3(\Lambda_{k-1}^*(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}), \mathbf{x}_{k-1}) \geq \tau\beta/4. \tag{5.35}
\]

By \( M_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}) \) denote the set of \( \mathbf{x}_{k-1} \in \overline{R}_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}) \) such that \( \mathbf{x}_{k-1} \) satisfies (5.28). Using (5.35), we obtain

\[
S_M := \frac{1}{|A_0|^2} \sum_{\mathbf{x}_0 \in A_0} \frac{1}{|A_1(\mathbf{x}_0)|^2} \sum_{\mathbf{x}_1 \in A_1(\mathbf{x}_0)} \cdots \sum_{\mathbf{x}_{k-1} \in M_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2})} g_3(\Lambda_{k-1}^*(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}), \mathbf{x}_{k-1}) \geq \tau\beta/8. \tag{5.36}
\]

Using (5.28), (5.33), (5.34) and (5.36), we get

\[
\text{ind}_{k+1}(\Lambda_{k+1})(g) \geq \frac{1}{|A_0|^2} \sum_{\mathbf{x}_0 \in A_0} \frac{1}{|A_1(\mathbf{x}_0)|^2} \sum_{\mathbf{x}_1 \in A_1(\mathbf{x}_0)} \cdots \sum_{\mathbf{x}_{k-1} \in M_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2})} \left\{ \frac{1}{\lambda_0} \sum_{\mathbf{x}_0 \in A_0} \frac{1}{|A_1(\mathbf{x}_0)|^2} \sum_{\mathbf{x}_1 \in A_1(\mathbf{x}_0)} \cdots \sum_{\mathbf{x}_{k-1} \in \Lambda_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2})} (g(\Lambda_{k-1}^*(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}), \mathbf{x}_{k-1})(1 + 2^{-15}r^2\beta^2\sigma^2) - 8\kappa) + \right. \\
\left. \sum_{\mathbf{x}_{k-1} \in \Lambda_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}) \setminus M_{k-1}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2})} (g(\Lambda_{k-1}^*(\mathbf{x}_0, \ldots, \mathbf{x}_{k-2}), \mathbf{x}_{k-1}) - 16\kappa) \right\} \\
\geq \text{ind}_{k-1}(\Lambda_{k-1})(g) + 2^{-15}r^2\beta^2\sigma^2 \left( \frac{\tau\beta}{8} \right) S_M - 24\kappa \\
\geq \text{ind}_{k-1}(\Lambda_{k-1})(g) + 2^{-24}r^4\beta^4\sigma^3 - 24\kappa \\
\geq \text{ind}_{k-1}(\Lambda_{k-1})(g) + 2^{-25}r^4\beta^4\sigma^3.
\]
In other words, for all \( k \geq 1 \), we have
\[
\text{ind}_{k+1}(\Lambda_{k+1})(g) \geq \text{ind}_{k-1}(\Lambda_{k-1})(g) + 2^{-25}\tau^4\beta^4\sigma^3. \tag{5.37}
\]

Since for any \( k \) we have \( \text{ind}_k(\Lambda_k)(g) \leq 1 \), it follows that the total number of steps of the algorithm does not exceed \( K_0 = 2^{30}\tau^{-4}\beta^{-4}\sigma^{-3} \).

Suppose that the algorithm stops at step \( K, K \geq 1, K \leq 2^{30}\tau^{-4}\beta^{-4}\sigma^{-3} \). We have
\[
\text{ind}_K(\Lambda_K, \overline{R}_K)(g_3) < \frac{\tau\beta}{4}. \tag{5.38}
\]
Using Lemma 5.4, we get
\[
\text{ind}_K(\Lambda_K)(g_1) \geq (\delta + \tau)\beta - 8\kappa K \geq (\delta + \frac{7\tau}{8})\beta.
\]
Using (5.38), we obtain
\[
\text{ind}_K(\Lambda_K, R_K)(g_1) \geq (\delta + \frac{3\tau}{8})\beta. \tag{5.39}
\]
The summation in (5.39) is taken over the sets \( \Lambda_K(x_0, \ldots, x_{K-1}) + \bar{y}, \) where \( \bar{y} \in R_K(x_0, \ldots, x_{K-1}) \).

Let \( E_K \) be the family of vectors \( \bar{y} \) such that \( \bar{y} \in E_K(x_0, \ldots, x_{K-1}) \), and \( R_K^{*} \) be the family of vectors \( \bar{y} \) such that \( \bar{y} \in R_K(x_0, \ldots, x_{K-1}) \), but \( \bar{y} \) does not belong to \( E_K(x_0, \ldots, x_{K-1}) \). We have
\[
\text{ind}_K(\Lambda_K, E_K)(g_1) < \frac{\tau\beta}{16} \text{ind}_K(\Lambda_K)(1) \leq \frac{\tau\beta}{16}. \tag{5.40}
\]
Combining (5.39), (5.40), we get
\[
\text{ind}_K(\Lambda_K, R_K^{*})(g_1) > (\delta + \frac{7\tau}{16})\beta. \tag{5.41}
\]
Suppose that for all \( \bar{y} \in R_K^{*}(x_0, \ldots, x_{K-1}) \), we have \( g_2(\Lambda_K(x_0, \ldots, x_{K-1}), \bar{y}) < (\delta + \tau/16) \). Then
\[
(\delta + \frac{\tau}{4})\beta < \text{ind}_K(\Lambda_K, R_K^{*})(g_1) \leq (\delta + \frac{\tau}{16}) \text{ind}_K(\Lambda_K, R_K^{*})(g_3) \leq
\]
\[
\leq (\delta + \frac{\tau}{16}) \text{ind}_K(\Lambda_K)(g_3). \tag{5.42}
\]
Using Lemma 5.4 once again, we obtain
\[
(\delta + \frac{\tau}{4})\beta < (\delta + \frac{\tau}{16}) \text{ind}_K(\Lambda_K)(g_3) \leq (\delta + \frac{\tau}{16})(\beta + 8\kappa K) \leq (\delta + \frac{\tau}{4})\beta
\]
with contradiction. Whence there exist vectors \( x_0, \ldots, x_{K-1}, \bar{y} \) such that \( g_2(\Lambda_K(x_0, \ldots, x_{K-1}), \bar{y}) \geq (\delta + \tau/16) \) and \( \bar{y} \in R_K(x_0, \ldots, x_{K-1}) \). We obtain the vector \( \bar{t} \), the sets \( E_1' = (E_1 - y_1) \cap \Lambda, E_2' = (E_2 - y_2) \cap \Lambda \) and the Bohr set \( \Lambda' \) which satisfy the conditions (1)–(3).

Let us estimate \( D \) and \( \varepsilon' \). At the each step of the algorithm the dimension of Bohr sets increases at most 1. Since the total number of steps does not exceed \( K_0 \), it follows that \( D < d + 2^{30}\tau^{-5}\beta^{-5}\sigma^{-3} \) and \( \varepsilon' \geq (2^{-20}\epsilon)\varepsilon \). Using Lemma 2.1 and (5.21), we obtain that the set \( \Lambda' \) is not empty. This completes the proof.

6. Proof of main result.
Let us put Theorem 4.1 and Proposition 5.1 together in a single proposition.
Proposition 6.1. Let $\Lambda = \Lambda(S, \varepsilon_0)$ be a Bohr set, $|S| = d$, and $\vec{s} = (s_1, s_2) \in G \times G$. Let $E_1, E_2$ be sets, $E_i = \beta_i|\Lambda|$, $i = 1, 2$, $\beta_1 \neq \beta_2$. Suppose $E = E_1 \times E_2$ is a subset of $(\Lambda + s_1) \times (\Lambda + s_2)$, $E_1, E_2$ are $(\alpha_0, 2^{-10\varepsilon_2})$-uniform subsets of $\Lambda + s_1$, $\Lambda + s_2$, respectively, $\alpha_0 = 2^{-20000\delta_{600}\beta_0^{48}}, \varepsilon = (2^{-100}\alpha_0^2)/(100d)$. Suppose that $A$ is a subset of $E$, $\delta E(A) = \delta$, and $A$ has no triples $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \neq 0$. Let

$$\log N \geq 2^{1000000}(2^{250000}\delta^{20000}\beta^{2000} + d)^3 \log \frac{1}{\delta \beta \varepsilon_0}. \tag{6.1}$$

Then there is a Bohr set $\tilde{\Lambda}$ and a vector $\tilde{y} = (y_1, y_2) \in G \times G$ with the following properties: there exist sets $E_1' \subseteq (E_1 - y_1 \cap \tilde{\Lambda}), E_2' \subseteq (E_2 - y_2 \cap \tilde{\Lambda})$ such that

1) Let $\rho_i(1) = \beta_i', \rho_i(2) = \beta_i'' \rho_i([\Lambda]$ and $\beta' \geq 2^{-1500}\beta^{100}$. 2) $E'_1, E'_2$ are $(\alpha'_0, 2^{-10}\varepsilon'_2)$-uniform, where $\alpha'_0 = 2^{-20000\delta_{600}\beta_0^{48}}, \varepsilon' = (2^{-100}\beta^{2000} + d)$. 3) For $\Lambda' = \Lambda(\tilde{S}, \varepsilon)$ we have $|\tilde{S}| = D$, and $\varepsilon \geq (2^{-100}\varepsilon_2)D \varepsilon_0$. 4) $\delta E'_1 \times E'_2(A) \geq \delta + 2^{-600\delta^{22}}$.

Proof of Theorem 1.4 Suppose that $A \subseteq G \times G, |A| = \delta N^2$ and $A$ has no triples $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \neq 0$.

The proof of Theorem 1.4 is a sort of an algorithm.

After the $i$th step of the algorithm a vector $\vec{s}_i = (s_i^{(1)}, s_i^{(2)})$ and sets : a regular Bohr set $\Lambda_i = \Lambda(S_i, \varepsilon_i)$, sets $E_i^{(1)} = \beta_i^{(1)}|\Lambda_i|, E_i^{(2)} = \beta_i^{(2)}|\Lambda_i|, \beta_i = \beta_i^{(1)}\beta_i^{(2)}$, $E_i = E_i^{(1)} \times E_i^{(2)}$.

The sets $\Lambda_i, E_i^{(1)}, E_i^{(2)}$ satisfy the following conditions

1) $\beta_i \geq 2^{-1500}\beta^{100}\beta_{i-1}$. 2) $E_i^{(1)}, E_i^{(2)}$ are $(\alpha_i^{(1)}, 2^{-10}\varepsilon_i^{(2)})$-uniform, $\alpha_i^{(1)} = 2^{-20000\delta_{600}\beta_0^{48}}, \varepsilon_i^{(2)} = 2^{-100}(\alpha_0^{(1)})^2/(100d_i)$. 3) $\Lambda_i = \Lambda(S_i, \varepsilon_i), |S_i| = d_i, d_i \leq 2^{500000}\delta^{20000}\beta_i^{100} + d_{i-1}, \varepsilon_i \geq (2^{-100}\varepsilon_i^{(2)})d_i \varepsilon_i^{1-1}$. 4) $\delta E_i(A') \geq \delta E_{i-1}(A') + 2^{-600\delta^{22}}$.

Proposition 6.1 allows us to carry the $(i + 1)$th step of the algorithm. By this Proposition there exists a new vector $\vec{s}_{i+1} = (s_{i+1}^{(1)}, s_{i+1}^{(2)}) \in G \times G$ and sets : a regular Bohr set $\Lambda_{i+1} = \Lambda(S_{i+1}, \varepsilon_{i+1})$, sets $E_{i+1}^{(1)} - s_i^{(1)} \subseteq \Lambda_{i+1}, E_{i+1}^{(2)} - s_i^{(2)} \subseteq \Lambda_{i+1}$, $E_{i+1} = E_{i+1}^{(1)} \times E_{i+1}^{(2)}$, which satisfy 1) — 4).

Put $S_0 = \{0\}, \Lambda_0 = \Lambda(S_0, 1)$ and $E_1 = E_2 = G, \beta_0 = 1$. Clearly, $E_1, E_2$ are $(2^{-20000\delta_{600}}, 2^{-10000\delta_{400}})$—uniform. Hence we have constructed zeroth step of the algorithm.

Let us estimate the total number of steps of our procedure. For an arbitrary $i$ we have $\delta E_i(A') \leq 1$. Using this and condition 4), we obtain that the total number of steps cannot be more then $2^{700\delta^{21}} = K$.

Condition 3) implies $\beta_i \geq (2^{-1500}\beta^{100}).$ Hence $d_i \leq (C_1\delta)^{-C_1i}$, where $C_1, C_1^i > 0$ are absolute constants.

To prove Theorem 1.4 we need to verify condition (6.1) at the last step of the algorithm. Condition (6.1) can be rewrite as

$$N \geq (C_2^i\delta)^{-C_2^i\delta^{21}} = \exp(\delta^{21}), \tag{6.2}$$
where $C_2', C_3', C_4', C' > 0$ are absolute constants. By assumption
\[
\delta \gg \frac{1}{(\log \log N)^{1/22}}
\]
and we get (6.2). Hence $A'$ has a triple $\{(k, m), (k + d, m), (k, m + d)\}$, where $d \neq 0$. This contradiction concludes the proof.

**Note.** Certainly, the constant 14 in Theorem 1.4 can be slightly decreased. Nevertheless, it is the author's opinion that this constant cannot be lowered to anything like 1 without a new idea.

Using the following lemma of B. Green (see e.g. [27] or [13]) one can obtain a corollary of Theorem 1.4 concerning subsets of $\{-N, \ldots, N\}^2$ without corners (see details in [27]).

**Lemma 6.1.** Let $N$ be a natural number. Suppose $A$ is a subset of $\{-N, \ldots, N\}^2$, $|A| = \delta(2N + 1)^2$, and $A$ has no triples $\{(k, m), (k + d, m), (k, m + d)\}$ with $d > 0$. Then there exists a set $A_1 \subseteq A$ such that
1) $|A_1| \geq \delta^4(2N + 1)^2/4$ and
2) $A_1$ has no triples $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \neq 0$.

**Corollary 6.1.** Let $\delta > 0$, and $N \gg \exp(\exp(-43))$. Let $A$ be a subset of $\{1, \ldots, N\}^2$ of cardinality at least $\delta N^2$. Then $A$ contains a triple $\{(k, m), (k + d, m), (k, m + d)\}$ with $d > 0$.

**References**


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