The geometry of Haldane limits

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Nonlinear σ-models

- Entered physics via low-energy QCD with the work of M.Gell-Mann and M.Levy (1960)
- Describe the scattering of Goldstone bosons in 4D, for example π-mesons in the case $\frac{SU(2)\times SU(2)}{SU(2)}$ $(u, d$ quarks)
- Renormalizable in 2D

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  Therefore provides a method for the exploration of target-space geometry
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- QFT requires regularization preserving the symmetries
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The goal / result

- The goal is to construct the spin chain with target space $\frac{U(N)}{U(n_1)\times\cdots\times U(n_m)}$.

The Hamiltonian

$$H = \sum_{i=1}^{L} \sum_{k=1}^{m-1} d_k \, \vec{S}_i \cdot \vec{S}_{i+k}$$

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- Aiming at an expression of the following form:

$$\mathcal{Z} = \int \prod_{t\in[0,1]} d\mu(z(t), \bar{z}(t)) \; \exp\left(-\mathcal{S}\right),$$

where $z \in \mathbb{CP}^{N-1}$ and

$$S = m \int_0^1 dt \sum_i \left(i \frac{\dot{z}_i \circ \bar{z}_i}{z_i \circ \bar{z}_i} + \beta \left| \frac{z_i \circ \bar{z}_{i+1}}{z_i \circ \bar{z}_i} \right|^2 \right)$$

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Elements of symplectic geometry

- Phase space $\mathcal{N}$ is a symplectic manifold.
- ω is a non-degenerate closed 2-form: $d\omega = 0$
- $G \circlearrowright \mathcal{N}$, the moment map $\mu : \mathcal{N} \to \mathfrak{g}$
- Equivariance:
  $$\mu(g \circ x) = Ad_g\, \mu(x) \equiv g\mu(x)g^{-1}$$
- Hamiltonians: $d\mu_a = i_{X_a}\omega, \quad a \in \mathfrak{g}$
- Simplest example: angular momentum

$$\mathcal{N} = \mathbb{R}^6,\ G = SO(3),\ \omega = d\vec{r} \wedge d\vec{p},\ \vec{L} = \vec{r} \times \vec{p}.$$

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- Let V be a representation of $U(N)$ with highest weight $\vec{\lambda}$.
- It can be built on the space of sections of a holomorphic fiber bundle

$$\mathcal{O}_1(\lambda_1)\otimes\cdots\otimes\mathcal{O}_N(\lambda_N)\to U(N)/U(1)^N$$

- In special cases the base can be reduced:

$$\frac{U(4)}{U(3)\times U(1)}=\mathrm{CP}^3$$

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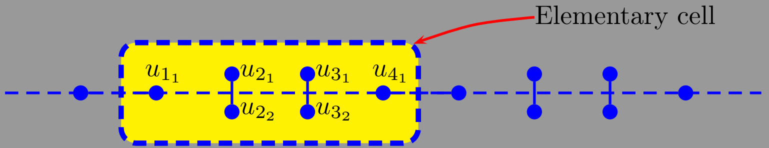
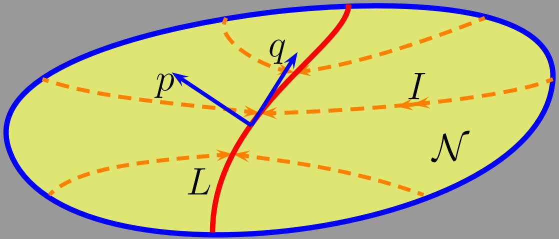
A picture of the spin chain



The spin chain for the coset $\frac{U(6)}{U(2)\times U(2)\times U(1)\times U(1)}$

The general setup

- The antiferromagnetic setup:



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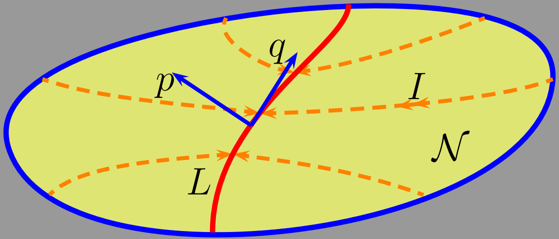

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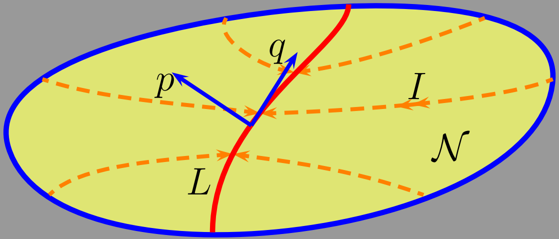

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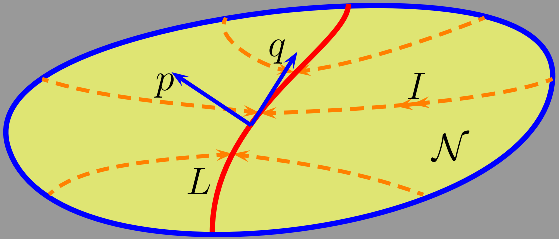

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The metric

- The metric in the normal directions to L: the Hessian $h_{ij} = \frac{\partial^2 I}{\partial x^i \, \partial x^j}$

The metric on L

$$g_{ij} = \omega_{im} \cdot \left[\left(\frac{\partial^2 I}{\partial x^2} \right)^{-1} \right]^{mn} \cdot \omega_{nj} = \omega_{im} \, h^{mn} \, \omega_{nj}$$

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The θ-term

- Construct the Lagrangian embedding

$$i : \mathcal{F}_{n_1,\cdots,n_m} \hookrightarrow G_{n_1} \times \ldots \times G_{n_m}$$

- Ingredients for the θ-term:

$$r_k = i^*\left(c_1(\mathcal{O}_{G_{n_k}}(1))\right), \quad \sum_{k=1}^m r_k = 0.$$

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$$\Omega = \frac{1}{m}\left(\sum_{k=1}^m k \cdot r_k\right)$$

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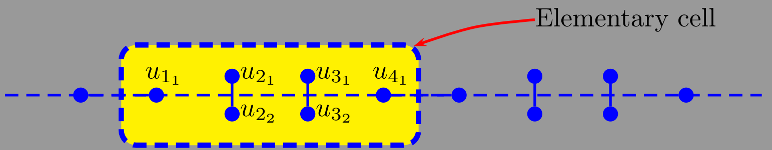
- Relation to the action of Weyl group on Schubert cells / Bruhat decomposition

The picture revisited



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Questions / Answers

- Constructed a spin chain for a σ-model with target space a flag manifold.
- Universal expressions for the metric and θ-term.
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Thank you!