

n -VALUED GROUPS: THEORY AND APPLICATIONS

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*To Victor Anatolievich Vassiliev,
an outstanding mathematician and a remarkable person,
on the occasion of his 50 ies birthday*

ABSTRACT.

We give a survey of the most important results in the theory of n -valued groups and some of their applications. Main directions of advanced research will be discussed.

We start with basic definitions. Further exposition follows a sequence of instructive examples originating from various branches of mathematics: algebra, analysis, representation theory, topology, and dynamical systems.

The contents is accessible to a broad audience.

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INTRODUCTION.

In various fields of research one encounters a natural multiplication on a space, say, X under which the product of a pair of points is a subset of X (e.g., a finite subset).

The literature on multivalued groups and their applications is very large and includes titles from 19th century, mainly in the context of hypergroups ([22, 25]).

In 1971, S. P. Novikov and the author (see [3]) introduced a construction, suggested by the theory of characteristic classes of vector bundles, in which the product of each pair of elements is *an n -multiset*, the set of n points with multiplicities.

This construction leads to the notion of *n -valued group*.

The condition of n -valuedness is in fact very restrictive, and initially it seemed that the supply of interesting examples of n -valued groups is not very rich.

Soon afterwards the author ([4, 5]) developed the theory of formal, or local, n -valued Lie groups, which appeared to be rich of contents and have found important applications ([6, 7, 8]). A number of results in the algebraic theory of multivalued formal groups was obtained by A. N. Kholodov in [23, 24].

Since 1993, E. Rees and the author collaborate on the topological and algebraic theory of n -valued groups ([9, 12, 14, 15]). The methods of the theory lead in particular to the solution of the problem about the rings of functions on symmetric powers of a space ([16, 18]).

The theory of n -valued groups has seminal connections with a number of classical and modern fields of research. The following directions of research will be in the focus of this paper:

- n -valued groups as deformations of ordinary groups;
- group algebras of n -valued groups as combinatorial algebras;
- n -Hopf algebras and their duals;
- algebraic representations of n -valued groups;
- representations of n -valued groups on graphs;
- integrable multivalued dynamical systems;
- functional-algebraic theory of symmetric products of spaces.

My talks at the Universities of Leeds, Liverpool and Manchester in autumn 2005 form a base of the present paper. I am grateful to Sir M. Atiyah, whose question initiated my talks, and to D. V. Leykin for useful discussion in the process of preparing them.

1. SYMMETRIC PRODUCT OF A SPACE.

If X is a topological space, let $(X)^n$ denote its n -fold symmetric product, i.e., $(X)^n = X^n/\Sigma_n$ where the symmetric group Σ_n acts by permuting the factors.

An n -**multiset** of a given space X is an unordered set of n points of X , taking multiplicities into account. Thus, $(X)^n$ is the topological space of all n -multisets of X .

Example. Let \mathbb{C} be the field of complex numbers. The spaces $(\mathbb{C})^n = \mathbb{C}^n/\Sigma_n$ and \mathbb{C}^n are identified using the map

$$\mathfrak{S} : (\mathbb{C})^n \rightarrow \mathbb{C}^n$$

whose components are given by

$$(z_1, z_2, \dots, z_n) \rightarrow e_r(z_1, z_2, \dots, z_n), \quad 1 \leq r \leq n,$$

where e_r is the r -th elementary symmetric polynomial.

The **projectivization** of the map \mathfrak{S} induces a homeomorphism between $(\mathbb{C}P^1)^n$ and the n -dimensional complex projective space $\mathbb{C}P^n$.

We would like to attract reader's attention to the following fact:

Let M^n be an n -dimensional (over \mathbb{R}) manifold without boundary, and let $k \geq 2$. Then the k th symmetric power $(M^n)^k$ is a manifold without boundary iff $n = 2$.

2. n -VALUED GROUP STRUCTURE.

An n -**valued multiplication** on X is a map

$$\mu : X \times X \rightarrow (X)^n.$$

$$\mu(x, y) = x * y = [z_1, z_2, \dots, z_n], \quad z_k = (x * y)_k.$$

We have the following natural generalizations of the classical axioms of group multiplication.

Associativity. The n^2 -sets:

$$\begin{aligned} & [x * (y * z)_1, x * (y * z)_2, \dots, x * (y * z)_n], \\ & [(x * y)_1 * z, (x * y)_2 * z, \dots, (x * y)_n * z] \end{aligned}$$

are equal for all $x, y, z \in X$.

Unit. An element $e \in X$ such that

$$e * x = x * e = [x, x, \dots, x]$$

for all $x \in X$.

Inverse. A map $\text{inv}: X \rightarrow X$ such that

$$e \in \text{inv}(x) * x \text{ and } e \in x * \text{inv}(x)$$

for all $x \in X$.

Definition. The map μ defines *an n -valued group structure* $\mathcal{X} = (X, \mu, e, \text{inv})$ on X if it is associative, has a unit and an inverse.

3. FIRST RESULTS.

Lemma 1. For each $m \in \mathbb{N}$, an n -valued group on X , with the multiplication μ , can be regarded as an mn -valued group by using as the multiplication the composition

$$X \times X \xrightarrow{\mu} (X)^n \xrightarrow{(D)^m} (X)^{mn},$$

where D is diagonal.

Lemma 2. Let $\mathcal{X} = (X, \mu_X, e_X, \text{inv}_X)$ and $\mathcal{Y} = (Y, \mu_Y, e_Y, \text{inv}_Y)$ be an n - and an m -valued group on X and Y , respectively. Then there is a natural structure of an nm -valued group on $X \times Y$, with the multiplication

$$\mu_{X \times Y}((x_1, y_1), (x_2, y_2)) = [(\mu_X(x_1, x_2))_i, (\mu_Y(y_1, y_2))_j], \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

the unit (e_X, e_Y) and the inverse map $\text{inv}_{X \times Y} = (\text{inv}_X, \text{inv}_Y)$.

Definition. A map $f: X \rightarrow Y$ is a **homomorphism** of n -valued groups if

$$f(e_X) = e_Y, \quad f(\text{inv}_X(x)) = \text{inv}_Y(f(x))$$

for all $x \in X$ and

$$\mu_Y(f(x_1), f(x_2)) = (f)^n \mu_X(x_1, x_2)$$

for all $x_1, x_2 \in X$.

A direct verification yields

Lemma 3. Let $f: X \rightarrow Y$ be a homomorphism of n -valued groups. Then

$$\text{Ker}(f) = \{x \in X \mid f(x) = e_Y\}$$

is an n -valued group.

4. A 2-VALUED GROUP STRUCTURE ON \mathbb{Z}_+ .

The following simple example of a 2-valued group arises in various constructions of the theory of multivalued groups and plays an important role in the applications.

Consider the semigroup of nonnegative integers \mathbb{Z}_+ .

Define the 2-valued multiplication

$$\mu: \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow (\mathbb{Z}_+)^2$$

by the formula

$$x * y = [x + y, |x - y|].$$

This multiplication endows \mathbb{Z}_+ with the structure of a 2-valued group with the unit $e = 0$. The inverse element is given by the identity map $\text{inv}(x) = x$.

The associativity axiom follows from the fact that the 4-multisets

$$[x + y + z, |x - y - z|, x + |y - z|, |x - |y - z||]$$

and

$$[x + y + z, |x + y - z|, |x - y| + z, ||x - y| - z|]$$

are equal for all nonnegative integers x, y, z .

5. ADDITIVE n -VALUED GROUP STRUCTURE ON \mathbb{C} .

Define the multiplication

$$\mu: \mathbb{C} \times \mathbb{C} \rightarrow (\mathbb{C})^n$$

by the formula

$$x * y = [(\sqrt[n]{x} + \epsilon^r \sqrt[n]{y})^n, \quad 1 \leq r \leq n],$$

where ϵ is a primitive n -th root of unity.

This multiplication endows \mathbb{C} with the structure of an n -valued group with the unit $e = 0$. The inverse element is given by the map $\text{inv}(x) = (-1)^n x$.

It is proved in [5] that, up to change of coordinates, there are only two different 2-valued formal group structures on \mathbb{C} , namely, the ordinary additive group and the one described above for $n = 2$. This fact is nontrivial (see Secs. 12–17 below).

The n -valued multiplication is described by the polynomials

$$p_n(z; x, y) = p_n = \prod_{k=1}^n (z - (\text{inv}(x) * \text{inv}(y))_k),$$

whence the product $x * y$ is defined by z -roots of the equation $p_n = 0$. It happens that $p_n(z; x, y)$ are x, y, z -symmetric polynomials with integral coefficients, e.g.,

$$p_1 = x + y + z, \quad p_2 = (x + y + z)^2 - 4(xy + yz + zx).$$

Here are the expressions of the polynomials defining the n -valued multiplication, for small values of n . Set

$$\mathbf{e}_1 = x + y + z, \quad \mathbf{e}_2 = xy + yz + zx, \quad \mathbf{e}_3 = xyz.$$

Then

$$p_3 = e_1^3 - 3^3 e_3,$$

$$p_4 = e_1^4 - 2^3 e_1^2 e_2 + 2^4 e_2^2 - 2^7 e_1 e_3,$$

$$p_5 = e_1^5 - 5^4 e_1^2 e_3 + 5^5 e_2 e_3,$$

$$p_6 = e_1^6 - 2^2 \cdot 3 e_1^4 e_2 + 2^4 \cdot 3 e_1^2 e_2^2 - 2^6 e_2^3 - 2 \cdot 3^4 \cdot 17 e_1^3 e_3 - 2^3 \cdot 3^4 \cdot 19 e_1 e_2 e_3 + 3^3 \cdot 19^3 e_3^2,$$

$$p_7 = e_1^7 - 5 \cdot 7^4 e_1^4 e_3 + 2 \cdot 7^6 e_1^2 e_2 e_3 - 7^7 e_2^2 e_3 + 7^8 e_1 e_3^2.$$

Now we arrive at the following natural questions:

- (1) What is the relationship between prime factors of n and prime factors of the coefficients of the polynomials p_n ?
- (2) How to distinguish the monomials that have zero coefficient? ¹
- (3) How to describe the Newton polytope of p_n ?

The answers on this questions have not been known yet. An important application of polynomials p_n we describe in Sec. 21.

6. COSET AND DOUBLE COSET GROUPS.

The following construction produces many n -valued groups.

Let G be a (1-valued) group with the multiplication μ_0 , the unit e_G , and $\text{inv}_G(u) = u^{-1}$.

Let A be a group with $\#A = n$ and $\varphi : A \rightarrow \text{Aut } G$ be a homomorphism to the group of automorphisms of G . Denote by X the quotient space $G/\varphi(A)$ of G by the action of the group $\text{Im } \varphi$, and denote by $\pi : G \rightarrow X$ the quotient map. Define the n -valued multiplication

$$\mu : X \times X \rightarrow (X)^n$$

by the formula

$$\mu(x, y) = [\pi(\mu_0(u, v^{a_i}))], \quad 1 \leq i \leq n, \quad a_i \in A],$$

where $u \in \pi^{-1}(x)$, $v \in \pi^{-1}(y)$ and v^a is the image of the action of $\varphi(a) \in \text{Aut } G$, $a \in A$ on G .

A direct verification leads to the following result:

Theorem 1. *The multiplication μ defines an n -valued group structure on the orbit space $X = G/\varphi(A)$, called the **coset group** of (G, A, φ) , with **the unit** $e_X = \pi(e_G)$ and **the inverse** $\text{inv}_X(x) = \pi(\text{inv}_G(u))$, where $u \in \pi^{-1}(x)$.*

In case of $\ker \varphi = 0$ we will identify A with $\varphi(A) \subset \text{Aut } G$.

Let $H \subset G$ be a subgroup, and let $\#H = n$. Denote by X the space of double cosets $H \backslash G / H$. Define the n -valued multiplication

$$\mu : X \times X \rightarrow (X)^n$$

¹I am grateful to the referee for this question.

by the formula

$$\mu(x, y) = \{Hg_1H\} * \{Hg_2H\} = [\{Hg_1hg_2H\} : h \in H].$$

Theorem 2. *The multiplication μ defines an n -valued group structure on the orbit space $X = H \backslash G / H$ called a **double coset group** of a pair (G, H) with **the unit** $e_X = \{H\}$ and **the inverse** $\text{inv}_X(x) = \{Hg^{-1}H\}$, where $x = \{HgH\}$.*

Each coset group $X = G/\varphi(A)$, where $\varphi : A \rightarrow \text{Aut } G$ be a homomorphism to the group of automorphisms of G , admits a double coset realization $X = A \backslash G' / A$, where G' is the semidirect product of the groups G and A with respect to the action of A on G by means of φ .

7. EXAMPLES OF THE COSET GROUPS.

Example 1. The 2-valued group on \mathbb{Z}_+ is the coset group of the pair $(\mathbb{Z}, \mathbb{Z}_2)$, where the generator of \mathbb{Z}_2 acts on \mathbb{Z} by multiplication by -1 .

Example 2. The additive n -valued group on \mathbb{C} is the coset group of the pair $(\mathbb{C}, \mathbb{Z}_n)$, where the generator of \mathbb{Z}_n acts on \mathbb{C} by multiplication by $\varepsilon = \exp \frac{2\pi i}{n}$.

Example 3. The multiplicative 2-valued group on \mathbb{C} . Let $G = \mathbb{C}^* = (\mathbb{C} \setminus \{0\})$. The automorphism $z \rightarrow z^{-1}$ generates the automorphism group A , $\#A = 2$. The space $X = \mathbb{C}^*/A$ is identified with \mathbb{C} by means of the map $\mathbb{C}^* \rightarrow \mathbb{C}$ given by $\pi(z) = \frac{1}{2}(z+z^{-1})$ and $\pi^{-1}(x) = x \pm (x^2 - 1)^{\frac{1}{2}}$, and we have

$$x * y = [xy + ((x^2 - 1)(y^2 - 1))^{\frac{1}{2}}, xy - ((x^2 - 1)(y^2 - 1))^{\frac{1}{2}}].$$

The right-hand side can be rewritten as the quadratic polynomial

$$z^2 - 2xyz + x^2 + y^2 - 1.$$

The unit is 1 and the inverse of each element is the element itself. Under the change of variables $x, y, z \mapsto x + 1, y + 1, z + 1$, the polynomial becomes

$$z^2 - 2(x + y + xy)z + (x - y)^2.$$

Example 4. Let G be the infinite dihedral group

$$G = \{a, b \mid a^2 = b^2 = e\}.$$

The exchange of a and b generates the automorphism group A , $\#A = 2$. Then

$$X = G/A = \{u_{2n}, u_{2n+1}\}, \quad n \geq 0,$$

where

$$\begin{aligned} u_{2n} &= \{(ab)^n, (ba)^n\}, \\ u_{2n+1} &= \{b(ab)^n, a(ba)^n\}. \end{aligned}$$

Then the 2-valued multiplication in the coset group is given by the formula

$$u_k * u_\ell = [u_{k+\ell}, u_{|k-\ell}|].$$

Thus X is isomorphic to the 2-valued group on \mathbb{Z}_+ .

Examples 1 and 4 show that the theory of coset groups cannot be reduced to the theory of pairs (G, A) , where A is a group of automorphisms of G , since distinct pairs (G_i, A_i) can lead to the same coset group.

Examples 1 and 3 describe coset groups in the class of coset groups of pairs (G, \mathbb{Z}_2) , where G is a commutative group and the generator of \mathbb{Z}_2 takes $g \in G$ to g^{-1} .

Another important example in this class is the continuous coset group of the pair (S^1, \mathbb{Z}_2) , where $S^1 = \{z \in \mathbb{C}, |z| = 1\}$. In this case $X = [-1, 1]$ and the projection $\pi: S^1 \rightarrow X$ is given by the map

$$\pi(z) = \frac{1}{2}(z + \bar{z}) = \cos \varphi, \quad 0 \leq \varphi \leq \pi.$$

The 2-valued multiplication

$$\mu: X \times X \longrightarrow (X)^2$$

has the form

$$\mu(x, y) = [xy \pm \sqrt{(1-x^2)(1-y^2)}],$$

where $|x| \leq 1$ and $|y| \leq 1$. The unit with respect to this multiplication is $1 \in X$, and $\text{inv}(x) = x$.

Note that the embedding $S^1 \subset \mathbb{C}^*$ leads to a homomorphism of 2-valued groups $f: [-1, 1] \rightarrow \mathbb{C}$ (see Example 3). The class of coset groups under discussion also contains the 2-valued coset groups of the pairs (T^k, \mathbb{Z}_2) , where T^k is the real k -torus $S^1 \times \cdots \times S^1$.

The Abelian manifolds, in particular, the Jacobians of algebraic curves of genus g give the important examples of such coset groups. In the case $g = 1$, we obtain a structure of a 2-valued group on the complex projective line $\mathbb{C}P^1$ (see details in Sec. 18), and in the case $g = 2$ we obtain the structure of a 2-valued group on Kummer surfaces.

Example 5. Let G be a finite group, $\#G = n$. Let $A = G$ act by inner automorphisms

$$g^a = a^{-1}ga, \quad g \in G, a \in A.$$

In this way we obtain the homomorphism $\varphi: A \rightarrow \text{Aut } G$ and a structure of an n -valued group on the set $X = G/\varphi(A)$ of conjugacy classes of the group G .

The first nontrivial example here is the coset group of the pair (G, G) , where $G = \Sigma_3$. We make use of the standard notation for conjugacy classes of the symmetric group Σ_n in terms of partitions of n .

For $n = 3$, we have $X = \{e = (1^3), x_1 = (21), x_2 = (3)\} = \{e, x_1, x_2\}$, which yields a 6-valued group structure on X , with the multiplication

$$\begin{aligned} x_1 * x_1 &= [e, e, e, x_1, x_1, x_1], \\ x_1 * x_2 &= x_2 * x_1 = [x_2, x_2, x_2, x_2, x_2, x_2], \\ x_2 * x_2 &= [e, e, x_1, x_1, x_1, x_1], \end{aligned}$$

the unit e and the identity map inv .

Note that this 6-valued group on three elements cannot be reduced to a group of smaller multiplicity.

Note also that the symmetric group Σ_n induces, for each $n \geq 2$, on the set of partitions $\{\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0), \sum_{i=1}^k \lambda_i = n\}$ of a given number n a structure of $(n!)$ -valued group, with inv being the identity map. A wide class of n -valued groups such that inv is not the identity map is constructed in the next section.

The construction of coset groups in Example 4 leads to the following class of multivalued groups.

Example 6. Let G be a group and let G_1 be a finite subgroup in G , $\#G_1 = n$. Thus we get the coset group of the pair $(G, \varphi(G_1))$, where G_1 acts on G by internal automorphisms. It can happen in this case that the resulting group is less than n -valued.

Let, for example, $g \in G$ be an element such that $g \notin Z$, and $g^2 \in Z$, where Z is the center of G . Then g generates a subgroup A , of order 2, in the group of internal automorphisms of G , and we obtain the 2-valued coset group of the pair (G, A) .

An important example in this class is given by the group $G = S^3 \simeq Sp(1)$ if one takes for g the purely imaginary quaternion unit j . It happens that the quotient space X of the sphere S^3 modulo the action of j by an internal automorphism is homeomorphic to the 3-sphere, i.e., we have a ramified covering $\pi : S^3 \rightarrow S^3$ (see [12]). Thus we obtain a nontrivial structure of a 2-valued group on S^3 .

Example 7. Consider the n -fold direct product G^n of a group G by itself. The group Σ_n acts on G^n by permuting the factors. Therefore, *for any group G , the symmetric product $(G)^n$ is endowed with the structure of an $(n!)$ -valued coset group.*

If G is commutative, with the operation $\mu(g', g'') = g' + g''$, then we have an $(n!)$ -valued group homomorphism

$$(\mu)^n : (G)^n \longrightarrow G, \quad [g_1, \dots, g_n] \longrightarrow g_1 + \dots + g_n,$$

where G is treated as an $(n!)$ -valued group with the diagonal operation $\mu(g', g'') = [g' + g'', \dots, g' + g'']$. In this way we obtain the $(n!)$ -valued group $\text{Ker}(\mu)^n$.

Take a smooth elliptic curve. It equips the torus T^2 with a commutative group structure. The construction above produces, for each n , a structure of an $(n!)$ -valued group on $(T^2)^n$, whence a *structure of an $(n!)$ -valued group on the complex projective space $\mathbb{C}P^{n-1} = \text{Ker}((\mu)^n : (T^2)^n \rightarrow T^2)$* . For $n = 2$, this yields a structure of a 2-valued group on $\mathbb{C}P^1$, which has been already mentioned in Example 4, and will be described explicitly in Sec. 18.

8. n -VALUED DEFORMATIONS OF A FINITE GROUP.

In this section, we describe a class of n -valued groups, which plays an important role both in the theory and applications of n -valued groups (see Secs. 9, 19).

Let G be a finite group, $\#G = m$. Denote by X the set of elements of G ,

$$X = \{x_0 = e, x_1, \dots, x_{m-1}\}.$$

Let $\ell \in \mathbb{N}$ and $k \in \mathbb{Z}_+$. Set $n = \ell + k(m - 1)$ and $X^0 = X \setminus e$. Using the group operation $x_i x_j \in G$, define the multiplication

$$x_i * x_j = \underbrace{[x_i x_j, \dots, x_i x_j]}_{\ell} \underbrace{[X^0, \dots, X^0]}_k, \quad i, j \neq 0.$$

Lemma 4. *The multiplication $*$ gives an n -valued group on X , with unit e and $\text{inv}(x_j) = x_j^{-1}$ if and only if $k = 0$ or $m < 4$.*

Denote this n -valued group by $X(G, \ell, k)$ and call it **the n -valued deformation of G** .

Clearly, $X(G, 1, 0) = G$. Also note that $X(G, r\ell, rk)$ is obtained from $X(G, \ell, k)$ by means of the diagonal map $(D)^r$.

For example, $X(\mathbb{Z}_3, 1, 1)$ is the 3-valued **coset** group of the pair (\mathbb{Z}_7, A) . Here A is generated by multiplication by 2 on \mathbb{Z}_7 and $\#A = 3$.

Every n -valued group on the set $X = \{x_0 = e, x_1\}$ has the form $X(\mathbb{Z}_2, l, k)$, where $l + k = n$.

In papers [28] and [29] (see also [31]), the notion of linear deformation of a discrete group that leads to multivalued groups has been introduced. The construction of such deformations makes use of differential-geometric properties of the variety of associative algebra structures on a given vector space, with a distinguished basis.

9. A FAMILY OF NON-COSET GROUPS.

Consider the $(2k + 1)$ -valued group $X(\mathbb{Z}_3, 1, k)$. The multiplication is given by the formulas

$$\begin{aligned} x_1 * x_1 &= \underbrace{[x_1, \dots, x_1]}_k \underbrace{[x_2, \dots, x_2]}_{k+1}, \\ x_1 * x_2 &= x_2 * x_1 = [e, \underbrace{[x_1, \dots, x_1]}_k \underbrace{[x_2, \dots, x_2]}_k], \\ x_2 * x_2 &= \underbrace{[x_1, \dots, x_1]}_{k+1} \underbrace{[x_2, \dots, x_2]}_k. \end{aligned}$$

Theorem 3 (see [11]). *Suppose that $4k + 3 = pq$, where q and p are prime numbers, $q > p$. Then the $(2k + 1)$ -valued group $X(\mathbb{Z}_3, 1, k)$ is **non-coset**.*

Note that any pair of **twin primes**, like $(3, 5), (5, 7), (11, 13), (17, 19), \dots$, defines a non-coset group in our family.

Доказательство. Suppose $X(\mathbb{Z}_3, 1, k) = G/A$ with $\#A = 2k + 1$. Since the $(2k + 1)$ -multiset $x_i * x_i$, $i = 1, 2$, does not contain e , the orbit $\pi^{-1}(x_i)$ does not contain simultaneously g and g^{-1} for all $g \in G \setminus e$. Since the $(2k + 1)$ -multiset $x_1 * x_2$ contains only one e , all elements of the orbit $\pi^{-1}(x_i)$ have multiplicity one. Thus

$$\#G = 1 + (2k + 1) + (2k + 1) = 4k + 3 = pq.$$

Since $q > p$, the q -subgroup of G , by the Sylow theorem, is normal and invariant with respect to all automorphisms of G .

Thus, $q - 1 = 2k + 1$, which implies that q is even. We obtain a contradiction. \square

Theorem 4 (S. Evdokimov, 2005). *The $(2k + 1)$ -valued group $X(\mathbb{Z}_3, 1, k)$ is a coset group if and only if $2k + 1 = p^s$, where p is a prime number.*

10. MULTIVALUED GROUPS ON SETS OF IRREDUCIBLE UNITARY REPRESENTATIONS OF GROUPS.

Let G be a finite group, $\#G = m$ and let $\rho_0, \rho_1, \dots, \rho_k$ be the set of all its irreducible unitary representations, where ρ_0 is the trivial one-dimensional representation. Consider the decomposition of tensor products of irreducible representations in direct sums of irreducible representations,

$$\rho_i \otimes \rho_j = \rho_i \rho_j = \sum_{l=0}^k a_{ij}^l \rho_l,$$

where a_{ij}^l is the multiplicity of the representation ρ_l in the product $\rho_i \rho_j$. We have $a_{ij}^l = a_{ji}^l$ and, as the classical theory implies, $a_{ij}^0 = 1$ if $\rho_j = \bar{\rho}_i$ and $a_{ij}^0 = 0$ otherwise.

Let d_l denote the dimension of ρ_l . Denote by n the least common multiple (LCM) of $d_i d_j$, $0 \leq i \leq j \leq k$, and introduce the set of integers

$$m_{ij}^l = n a_{ij}^l \frac{d_l}{d_i d_j}.$$

By construction, for all i, j , where $0 \leq i, j \leq k$, we have

$$\sum_{l=0}^k m_{ij}^l = n.$$

Set $x_l = \frac{1}{d_l} \rho_l$ and consider the set $X = \{x_0, \dots, x_k\}$.

Theorem 5. *The tensor product of representations defines on X a structure of an n -valued group with the product $\mu : X \times X \rightarrow (X)^n$, where $\mu(x_i, x_j) = x_i * x_j$ is the n -multiset containing the element x_l with the multiplicity m_{ij}^l , the element x_0 is the unit, and the inverse $\text{inv} : X \rightarrow X$ is given by the complex conjugation map, i.e., $\text{inv}(x_l) = \bar{x}_l$, where $\bar{x}_l = \frac{1}{d_l} \bar{\rho}_l$.*

Example. $G = \Sigma_3$. There are irreducible representations ρ_0, ρ_1, ρ_2 , of dimensions $d_0 = 1$, $d_1 = 1$, and $d_2 = 2$, respectively, with the tensor product table

$$\begin{aligned} \rho_0 \rho_l &= \rho_l, \quad l = 0, 1, 2, \\ \rho_1^2 &= \rho_0, \quad \rho_1 \rho_2 = \rho_2, \quad \rho_2^2 = \rho_0 \oplus \rho_1 \oplus \rho_2. \end{aligned}$$

Therefore, $n = d_2^2 = 4$, and we obtain on the set $X = \{x_0 = \rho_0, x_1 = \rho_1, x_2 = \frac{1}{2} \rho_2\}$ a 4-valued group with the multiplication

$$x_1 * x_1 = [x_0, x_0, x_0, x_0], \quad x_1 * x_2 = [x_2, x_2, x_2, x_2], \quad x_2 * x_2 = [x_0, x_1, x_2, x_2],$$

the unit $e = x_0$, and the identity map inv .

Note that in this case the 4-valued group structure cannot be replaced by a less valued structure.

It is well known that associating to each irreducible representation ρ_l its character $\chi(\rho_l)$, one can identify the set of irreducible representations of a group G with the set of conjugacy classes of G . The classical theory implies the following statement.

Let ρ_λ be an irreducible unitary representation of the symmetric group Σ_n corresponding to the partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$, $\sum_{i=1}^k \lambda_i = n$ of n . Then the dimension of ρ_λ is

$$d_\lambda = \frac{n!}{l(\lambda)!} \Delta(l(\lambda)),$$

where $l(\lambda) = (l_1, \dots, l_k)$, $l_i = \lambda_i + k - i$, $i = 1, \dots, k$, $l(\lambda)! = l_1! \dots l_k!$, and $\Delta(l(\lambda)) = \prod_{i < j} (l_i - l_j)$.

Hence the tensor product of irreducible representations of the group Σ_n endows the set $\{\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0), \sum_{i=1}^k \lambda_i = n\}$ with a structure of an m -valued group, where $m = \text{LCM}\{d_\lambda d_\mu\}$.

In the case of a commutative group G , the identification of irreducible unitary representations with the conjugacy classes of G corresponds to the Pontryagin duality. For a general finite group G , with $\#G = m$, we obtain on these isomorphic sets structures of two distinct multivalued groups (one being n -valued, the other one m -valued, where $n = \text{LCM}\{d_i d_j\}$). For example, for $G = \Sigma_3$ one has $m = 6$ and $n = 4$.

This construction forms a base of the construction of a dual multivalued group for a given noncommutative finite group, see [17].

To give a flavor of what does the same construction produce for compact Lie groups, consider the case $G = SU(2)$.

Here is the list of all its unitary irreducible representations: ρ_{2l+1} , $l = 0, \frac{1}{2}, 1, \dots, \frac{s}{2}, \dots$, where $\dim \rho_{2l+1} = 2l + 1$. The character of the representation ρ_{2l+1} is

$$\chi_{2l+1} = \sum_{k=-l}^l \exp(-ikt), \quad l = 0, \frac{1}{2}, 1, \dots, \frac{s}{2}, \dots$$

The knowledge of the characters yields

$$\rho_{2l_1+1} \rho_{2l_2+1} = \sum_{k=|l_1-l_2|}^{l_1+l_2} \rho_{2k+1}.$$

There are the Adams operators Ψ^k on the ring of virtual representations of any group G . The following formula describes the characteristic property of this operators:

$$\chi(\Psi^k(\theta))g = \chi(\theta)g^k, \quad g \in G,$$

where $\chi(\theta)$ is the character of a representation $\theta : G \rightarrow GL(n, \mathbb{C})$ (see [1]).

Applying successively the Adams operators Ψ^k , $k = 1, 2, \dots$, where Ψ^1 is the identity operator, let us introduce the sequence of virtual representations $\psi_k = \Psi^k \rho_2$, $k = 1, 2, \dots$, and set $\psi_0 = \rho_1$. We have $\chi(\psi_k) = \exp(i\frac{k}{2}t) + \exp(-i\frac{k}{2}t)$. Therefore, $\psi_1 = \rho_2$ and $\psi_k = \rho_{k+1} - \rho_{k-1}$, $k > 1$. Note that $\dim \psi_0 = 1$ and $\dim \psi_k = (k+1) - (k-1) = 2$, $k \geq 1$. Put $x_0 = \psi_0$, $x_k = \frac{1}{2}\psi_k$, $k = 1, 2, \dots$, and consider the set $X = (x_0, x_1, \dots, x_k, \dots)$.

Lemma 5. *The tensor product of irreducible representations of the group $SU(2)$ endows X with the multiplication $\mu(x_k, x_m) = [x_{k+m}, x_{|k-m|}]$ with the unit x_0 and the identity map inv , whence a structure of a 2-valued group isomorphic to the 2-valued group \mathbb{Z}_+ .*

For a general compact Lie group G , we obtain an n -valued group, where n is the order of the Weyl group of G .

11. INVOLUTIVE AND SINGLY GENERATED MULTIVALUED GROUPS.

Let x, y and z be elements of some n -valued group. Denote by $m_{x,y}^z$ the multiplicity of z in the multiset $x * y$. For example, $m_{x,e}^x = n$, where e is the unit of the group.

Definition. A multivalued group $\mathcal{X} = (X, *, e, \text{inv})$ is said to be *involutive* if the map $\text{inv} : X \rightarrow X$ is an involution satisfying the following conditions:

- (1) $m_{x,y}^e \neq 0$ iff $y = \text{inv}x$;
- (2) $m(x) = m(\text{inv}x)$, where $m(x) = m_{x,\text{inv}x}^e$.

Note that the condition (1) implies that the map inv , for a given multivalued multiplication, is unique, whence

$$\text{inv}(x * y) = (\text{inv}y) * (\text{inv}x).$$

Coset and double coset groups are examples of multivalued involutive groups. The multivalued groups $X(G, l, k)$ (see Sec. 8) also are involutive. The multivalued group on the set of irreducible unitary representations of a finite group (see the previous section) also is involutive. On the other hand, examples of n -valued groups that are not involutive also are known (see [15]).

Definition. A multivalued group $\mathcal{X} = (X, *, e, \text{inv})$ is said to be *singly generated* if there is an element $x \in X$ such that each element $y \in X$ belongs to the multiset $x^k = x * \dots * x$ for some $k > 1$.

For ordinary groups, the only singly generated groups are \mathbb{Z} and its quotients, the cyclic groups \mathbb{Z}_m . The variety of singly generated multivalued groups is much wider.

Singly generated multivalued groups define integrable multivalued dynamical systems with discrete time, in terms of [27]. Lately, these dynamical systems are extensively studied, hence the problem of constructing singly generated multivalued groups and their action on spaces is an actual one.

Recall the Burnside theorem (see [21]):

Let ρ be an irreducible faithful representation of a finite group G . Then each irreducible representation of G enters the decomposition of a power $\rho^k = \rho \otimes \dots \otimes \rho$ into the sum of irreducible summands, for some k .

Due to the Burnside theorem, the following result yields a large family of singly generated multivalued groups.

Corollary 1. *If a finite group G possesses a **faithful** irreducible representation, then the multivalued group on the set of its irreducible representations is **singly generated**.*

For example, for $G = \Sigma_3$ the 2-dimensional irreducible representation ρ_2 is faithful, and the 4-valued group on $X = (e, x_1, x_2)$ described above is singly generated, namely, $x_2 * x_2 = (e, x_1, x_2, x_2)$.

The symmetric group Σ_n , $n \geq 2$, has an irreducible faithful representation ρ_λ , $\lambda = (n - 1, 1)$, of dimension $(n - 1)$. Hence the m -valued group on the set of unitary representations of the group Σ_n described above is **singly generated**.

In Sec. 19, we give a new definition of integrability of multivalued dynamics with discrete time taking action of multivalued groups for the base. The motivation for these results is based on the results in [10].

12. FORMAL 2-VALUED GROUPS.

The material of Secs. 12–17 is based on the author’s results about classification of one-dimensional formal 2-valued groups over \mathbb{Q} -rings (see [5, 6]). Using the method of generalized shift operators developed in these papers, A. N. Kholodov obtained in [24] a classification of multidimensional formal 2-valued groups over \mathbb{Q} -rings. Note also that the author constructed a universal formal 2-valued group having important applications in the theory of cobordisms of manifolds endowed with a symplectic ($Sp(N)$) structure in the stable tangent bundle. This result requires development of methods of homological algebra.

The problem of classification of algebraic manifolds equipped with an n -valued group structure is closely related to several urgent problems of algebraic geometry and the theory of integrable systems (see [19, 20]).

In Secs. 12–17, we work over the field of complex numbers and our main goal is to attract the attention of experts in algebraic geometry, the theory of integrable systems and singularity theory.

Consider the equation

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0$$

with respect to z , where $\Theta_1(x, y)$ and $\Theta_2(x, y)$ are formal series in (x, y) near $(0, 0) \in \mathbb{C} \times \mathbb{C}$.

Suppose:

- (1) $\Theta_1(x, 0) = 2x$ and $\Theta_2(x, 0) = x^2$.
- (2) There exists a series $\varphi(x)$ such that

$$\Theta_2(x, \varphi(x)) = 0.$$

- (3) Let $z_+(x, y)$ and $z_-(x, y)$ be symbols such that

$$z_+(x, y) + z_-(x, y) = \Theta_1(x, y),$$

$$z_+(x, y) \cdot z_-(x, y) = \Theta_2(x, y).$$

Then for any symmetric formal series $\Phi(t_1, t_2) = \Phi(t_2, t_1) = \widehat{\Phi}(t_1 + t_2, t_1 t_2)$ we can define $\Phi(z_+(x, y), z_-(x, y))$ as the formal series $\widehat{\Phi}(\Theta_1(x, y), \Theta_2(x, y))$.

Set $X_{\pm} = z_{\pm}(u, v)$ and $Y_{\pm} = z_{\pm}(v, w)$.

The following equalities of formal series in u, v, w hold:

$$\Theta_1(u, Y_+) + \Theta_1(u, Y_-) = \Theta_1(X_+, w) + \Theta_1(X_-, w);$$

$$\begin{aligned} \Theta_2(u, Y_+) + \Theta_2(u, Y_-) + \Theta_1(u, Y_+)\Theta_1(u, Y_-) &= \\ &= \Theta_2(X_+, w) + \Theta_2(X_-, w) + \Theta_1(X_+, w)\Theta_1(X_-, w); \end{aligned}$$

$$\begin{aligned} \Theta_2(u, Y_+)\Theta_1(u, Y_-) + \Theta_1(u, Y_+)\Theta_2(u, Y_-) &= \\ &= \Theta_2(X_+, w)\Theta_1(X_-, w) + \Theta_1(X_+, w)\Theta_2(X_-, w); \end{aligned}$$

$$\Theta_2(u, Y_+)\Theta_2(u, Y_-) = \Theta_2(X_+, w)\Theta_2(X_-, w).$$

Definition. When the conditions (1)–(3) are satisfied, we say that the equation

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0$$

defines a *formal 2-valued group law on \mathbb{C}* .

13. RELATION TO THE CLASSICAL FORMAL GROUP LAW.

The direct problem. Suppose that the equation

$$z - F(x, y) = 0,$$

where $F(x, y)$ is a formal series, defines a classical formal group law.

This means that the series $F(x, y)$ satisfies the conditions:

$$(1) F(x, 0) = x.$$

$$(2) F(x, F(y, z)) = F(F(x, y), z).$$

Then the equation

$$z^2 - 2F(x, y)z + F(x, y)^2 = 0$$

defines a formal 2-valued group law on \mathbb{C} .

The back problem. Suppose the equation

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0 \tag{*}$$

defines a formal 2-valued group law on \mathbb{C} .

Theorem 6. If the series $z = F(x, y)$ is a solution of (*) satisfying the condition $F(x, 0) = x$, then

$$\Theta_1(x, y) = 2F(x, y), \quad \Theta_2(x, y) = F(x, y)^2,$$

and the equation

$$z - F(x, y) = 0$$

defines the classical formal group law.

14. THE TYPE OF A 2-VALUED GROUP.

Consider the 2-valued group law

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0.$$

Using the axioms (1) and (3), we derive from the definition of formal 2-valued groups the following important fact:

Lemma 6. $\frac{\partial^2 \Theta_2(x, y)}{\partial x \partial y} \Big|_{(0,0)} = \pm 2.$

Definition. A formal 2-valued group is called a **1-st type group** when

$$\frac{\partial^2 \Theta_2(x, y)}{\partial x \partial y} \Big|_{(0,0)} = -2;$$

and is called a **2-nd type group** otherwise.

Definition. *The elementary 2-valued group of 1-st type* is defined by the equation:

$$z^2 - 2(x + y)z + (x - y)^2 = 0.$$

It is precisely the equation $p_2 = e_1^2 - 4e_2 = 0$ with the solutions

$$z_{\pm}(x, y) = (\sqrt{x} \pm \sqrt{y})^2.$$

The elementary 2-valued group of 2-nd type is defined by the equation:

$$(z - (x + y))^2 = 0.$$

15. THE STRONG ISOMORPHISM.

Consider the 2-valued group laws

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0, \tag{A}$$

$$z^2 - \widehat{\Theta}_1(x, y)z + \widehat{\Theta}_2(x, y) = 0. \tag{B}$$

Definition. The group laws (A) and (B) are **strongly isomorphic** if there exists a power series

$$\psi(x) = x(1 + O(x))$$

(a *regular change of coordinate*) such that

$$z_{\pm}(x, y) = \psi^{-1}(\widehat{z}_{\pm}(\psi(x), \psi(y))),$$

where $\psi^{-1}(\psi(x)) = x.$

Lemma 7. *Strong isomorphism preserves the type.*

Note that an *irregular coordinate change*

$$\Psi(x) = x^2(1 + O(x))$$

takes a 1-st type group law to a group law of the 2-nd type.

16. THE EXPONENT AND THE LOGARITHM OF A 2-VALUED GROUP.

Definition. The series

$$\psi(x) = x(1 + O(x))$$

defining a strong isomorphism of a 2-valued group with the elementary group is called *the logarithm of the group*.

The inverse series

$$\psi^{-1}(x) = x(1 + O(x)),$$

that is $\psi^{-1}(\psi(x)) = x$, is called *the exponent of the group*.

An application of the techniques of generalized shift operators allows one to obtain the following result:

Theorem 7. *Each 2-valued formal group on \mathbb{C} has the logarithm.*

Sketch of a proof. (For details see [6].)

Consider the differential operator

$$\mathcal{D}_x = \alpha_1(x) \frac{d}{dx} + \alpha_2(x) \frac{d^2}{dx^2},$$

where $\alpha_1(x), \alpha_2(x) \in \mathbb{C}[[x]]$ and $\alpha_1(0) = 1, \alpha_2(0) = 0$.

Lemma 8. *Let $\varphi(x) \in \mathbb{C}[[x]]$. Then the problems*

$$\begin{aligned} \mathcal{D}_x u(x, y) &= \mathcal{D}_y u(x, y) & \text{and} & & \mathcal{D}_x u(x) &= 1 \\ u(x, 0) &= \varphi(x) & & & u(0) &= 0 \end{aligned}$$

have solutions in formal series

$$u(x, y) \in \mathbb{C}[[x, y]] \quad \text{and} \quad u(x) \in \mathbb{C}[[x]],$$

respectively, iff the number $-\frac{1}{\alpha_2'(0)}$ is not an integer. Moreover, the solutions are uniquely determined by the initial conditions $u(x, 0) = \varphi(x)$ and $u(0) = 0$, respectively.

Pay attention to the fact that both problems are set for a differential operator of the second order whose coefficient of the second derivative vanishes at $x = 0$.

17. GENERALIZED SHIFT.

Suppose $-\frac{1}{\alpha_2'(0)} \notin \mathbb{N}$. Then the linear map

$$T_x^y: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x, y]]$$

is defined by the formula $T_x^y \varphi(x) = u(x, y)$, where the series $u(x, y)$ is the solution of the problem

$$\begin{aligned} \mathcal{D}_x u(x, y) &= \mathcal{D}_y u(x, y) \\ u(x, 0) &= \varphi(x). \end{aligned}$$

Lemma 9. *The operator T_x^y is a generalized shift, that is:*

- (1) the operator T_x^0 is **identity**;
- (2) the operator

$$T_y^z T_x^y - T_x^y T_x^z: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x, y, z]]$$

is zero (**associativity condition**).

Consider the equation

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0. \quad (*)$$

It follows from the axioms of 2-valued group that for (*) to define a 2-valued group it is **necessary** that

$$\begin{aligned} \Theta_1(x, y) &= 2(x + y) + \text{higher terms}, \\ \Theta_2(x, y) &= (x \pm y)^2 + \text{higher terms}. \end{aligned}$$

Define formal series $P_k(x, y)$, $k = 0, 1, \dots$, by the generating function

$$\sum_{k \geq 0} \frac{P_k(x, y)}{t^{k+1}} = \frac{2t - \Theta_1(x, y)}{2(t^2 - \Theta_1(x, y)t + \Theta_2(x, y))}.$$

Introduce a linear map

$$L_x^y: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x, y]]$$

by the formula $L_x^y x^k = P_k(x, y)$.

Take the differential operator \mathcal{D}_x with $\alpha_1(x) = \phi_1(x)/2$ and $\alpha_2(x) = \phi_2(x)/8$, where

$$\begin{aligned} \phi_1(x) &= \left. \frac{\partial \Theta_1(x, y)}{\partial y} \right|_{y=0}, \\ \phi_2(x) &= \left. \frac{\partial \sigma(x, y)}{\partial y} \right|_{y=0}, \end{aligned}$$

where $\sigma(x, y) = \Theta_1(x, y)^2 - 4\Theta_2(x, y)$ is the discriminant of the equation (*). By the above necessary conditions, we have

$$\begin{aligned} \alpha_1(0) &= 1, \quad \alpha_2(0) = 0, \\ \alpha_2'(0) &= \begin{cases} 2, & \text{1-st type} \\ 0, & \text{2-nd type} \end{cases} \end{aligned}$$

Thus the generalized shift T_x^y is well defined, by means of this \mathcal{D}_x .

Theorem 8. *If*

$$\begin{aligned} \Theta_1(x, y) &= 2(x + y) + \text{higher terms}, \\ \Theta_2(x, y) &= (x \pm y)^2 + \text{higher terms}, \end{aligned}$$

then $z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0$ defines a 2-valued group if and only if

$$L_x^y x^k = T_x^y x^k$$

for $k = 1, 2, 3, 4$.

Consider the 1-st type 2-valued group law

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0.$$

Lemma 10.

$$\phi_2(x) = 8 \int_0^x \phi_1(x) dx,$$

and $\phi_2(0) = 0$, $\phi_2'(0) = 16$.

Introduce $\Phi(x) = \frac{\phi_2(x)}{16x} = 1 + O(x)$.

Theorem 9. *The formula*

$$\psi(x) = \left(\int_0^{\sqrt{x}} \frac{dt}{\sqrt{\Phi(t^2)}} \right)^2$$

defines the series $\psi(x) = x(1 + O(x))$ such that

$$\mathcal{D}\psi(x) = 1.$$

The series $\psi(x)$ is ***the logarithm***.

Consider the 2-nd type 2-valued group law

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0.$$

Theorem 10. *The formula*

$$\psi(x) = 2 \int_0^x \frac{dt}{\phi_1(t)}$$

defines the series $\psi(x) = x(1 + O(x))$ such that

$$\mathcal{D}\psi(x) = 1.$$

The series $\psi(x)$ is ***the logarithm***.

18. ALGEBRAIC 2-VALUED GROUP STRUCTURES ON \mathbb{C} AND $\mathbb{C}P^1$.

Consider the series

$$\varphi(x) = -\frac{1}{\wp(\sqrt{-x}; g_2, g_3)} = x(1 + O(x)),$$

where $\wp(z; g_2, g_3)$ is the Weierstrass elliptic function with the invariants g_2 and g_3 .

Theorem 11 (see [8]). *The series $\varphi(x)$ is ***the exponent*** of the 2-valued group on \mathbb{C} defined by the equation*

$$\left(x + y + z + \frac{g_2}{4}xyz \right)^2 - (4 + g_3xyz)(xy + yz + zx) = 0.$$

In homogeneous coordinates, we obtain, by Theorem 10, the following result in [12]:

Theorem 12. *The map $\mu : \mathbb{C}P^1 \times \mathbb{C}P^1 \rightarrow \mathbb{C}P^2$ defined by*

$$\begin{aligned} \mu((x_0 : x_1), (y_0 : y_1)) &= \\ &= ((x_0y_0 - k^2x_1y_1)^2 : -2((x_0y_1 + x_1y_0)(x_0y_0 + k^2x_1y_1) - 4\delta x_0y_0x_1y_1) : (x_1y_0 - x_0y_1)^2) \end{aligned}$$

defines a 2-valued group structure on $\mathbb{C}P^1$ if and only if $k(k^2 - \delta^2) \neq 0$.

Note that in [15], a classification of all algebraic 2-valued group structures on $\mathbb{C}P^1$ is obtained.

19. ACTIONS AND INTEGRABLE n -VALUED DYNAMICS.

Definition. An n -valued group X *acts on a space* Y if there is a mapping

$$\phi : X \times Y \rightarrow (Y)^n,$$

also denoted $x \circ y = \phi(x, y)$, such that the two n^2 -multisets of Y

$$x_1 \circ (x_2 \circ y) \quad \text{and} \quad (x_1 * x_2) \circ y$$

are equal for all $x_1, x_2 \in X$ and $y \in Y$; and also

$$e \circ y = [y, y, \dots, y]$$

for all $y \in Y$.

Example. An n -valued group X acts on itself by left shifts,

$$\varphi_l(x, y) = x * y,$$

and an involutive group acts on itself also by right shifts:

$$\varphi_r(x, y) = y * \text{inv}(x).$$

It is well known that treating an action of a group G on a space V as a dynamical system leads to fruitful investigations in ergodic theory and geometry. Among the directions of these investigations, the study of dynamical systems with discrete time whose evolution is described by a map of V into itself plays a distinguished role. It is natural to apply the theory of n -valued groups to multivalued analogues of these systems.

Definition. An *n -valued dynamics T with discrete time* on a space Y is a continuous map $T : Y \rightarrow (Y)^n$.

Thus, if we consider Y as a space of states, then an n -valued dynamics $T : Y \rightarrow (Y)^n$ determines possible states $T(y) = [y_1, \dots, y_n]$ at the moment $(t + 1)$ as functions of the state y of the system at the moment t .

Definition. An n -valued dynamics T with discrete time on a space Y is said to be *integrable* by means of an n -valued group X with a single generator a if the embedding

$i : Y \rightarrow X \times Y$, $i(y) = (a, y)$, extends to an action φ of the n -valued group X on Y , i.e., the triangle

$$\begin{array}{ccc} Y & \xrightarrow{T} & (Y)^n \\ \downarrow i & \nearrow \varphi & \\ X \times Y & & \end{array}$$

is commutative.

A large class of n -valued dynamics integrable in the sense of the previous definition is produced by the following construction of *n -valued dynamics on a triangulation of a manifold*:

Let M^n be a closed manifold. Pick a triangulation of M^n and take for Y the set of all n -dimensional simplices of this triangulation. Define the n -valued dynamics $T : Y \rightarrow (Y)^{n+1}$ which associates to an n -dimensional simplex σ^n the set of n -dimensional simplices $T\sigma^n = [\sigma_1^n, \dots, \sigma_{n+1}^n]$, $\sigma_i^n \neq \sigma^n$, $i = 1, \dots, n+1$, having a nonempty intersection with σ^n (the set of all its neighbors).

Example. Pick the triangulation of the n -dimensional sphere S^n as the boundary of an $(n+1)$ -dimensional simplex. There are $(n+2)$ simplices of dimension n in this triangulation, i.e., Y consists of $(n+2)$ points. A direct verification shows that the corresponding dynamics $T : Y \rightarrow (Y)^{n+1}$ is integrable by means of the singly generated group $X(\mathbb{Z}_2, 1, n)$ (see Sec. 8).

We devote further papers to the study of triangulations of M^n in terms of this $(n+1)$ -valued dynamics.

20. THE COSET AND DOUBLE COSET CONSTRUCTIONS OF THE ACTIONS.

Let G be a certain (ordinary) group and let A be a finite group of automorphisms of G , $\#A = n$. Suppose that G and A act on some space V in such a way that

$$a(g(v)) = a(g)(a(v)), \text{ where } a \in A, g \in G, v \in V.$$

In other words: the action of G on V is *equivariant* with respect to the action of A on V and the diagonal action of A on $G \times V$.

Take for X the coset group G/A and take for Y the space of orbits V/A . Then *there is a natural action of the n -valued group X on the space Y .*

Let G be a group and let $H \subset G$ be a subgroup, $\#H = n$. Suppose G acts on a space V . Take for X the double coset group $H \backslash G / H$ and take for Y the space of orbits V/H . Then X acts on Y according to the formula

$$\{HgH\}\{Hv\} = [H(gh)v : h \in H].$$

Definition. A representation of an n -valued group X in an algebra L is a map

$$\rho : X \longrightarrow L$$

such that $\rho(e) = 1$ and

$$\text{Av}\rho(x * y) = \rho(x)\rho(y),$$

where $\text{Av}\rho(x * y)$ denotes the average value of ρ on the set $x * y = [z_1, \dots, z_n]$, i.e.

$$\text{Av}\rho(x * y) = \frac{1}{n} \sum_{i=1}^n \rho(z_i).$$

Example. Let $P(S^1)$ be the algebra of trigonometric polynomials on the circle and let $L \subset P(S^1)$ be the subalgebra of even polynomials, with the additive basis $\{\cos k\theta, k \geq 0\}$. Then the map

$$\rho : \mathbb{Z}_+ \longrightarrow L \quad : \quad \rho(k) = \cos k\theta$$

defines a representation of the 2-valued group \mathbb{Z}_+ in L , i.e.,

$$\rho(k)\rho(l) = \text{Av}\rho(k * l).$$

An important special case is when L is the algebra of endomorphisms of a vector space V . In this case one has **a linear representation** of multivalued group X .

Let $\varphi : X \times Y \rightarrow (Y)^n$ be an action of the n -valued group X on Y . Consider the vector space $V = \mathbb{C}(Y)$ over \mathbb{C} spanned by Y . Recall that if Y is an infinite set, then the vectors in $\mathbb{C}(Y)$ are linear combinations of finitely many points in Y . Then the linear representation

$$\rho : X \longrightarrow \text{End}(V),$$

such that the linear operator $\rho(x)$ acts on the basic vectors $y \in Y$ according to the rule

$$\rho(x)y = \frac{1}{n}(y_1 + \dots + y_n),$$

where $\varphi(x, y) = x \circ y = [y_1, \dots, y_n]$, is well defined.

Definition. The **group algebra** of an n -valued group X is the vector space $\mathbb{C}(X)$ endowed with the multiplication that has the following form on the basic vectors $x \in X$:

$$x_1 x_2 = \frac{1}{n}(z_1 + \dots + z_n),$$

where $x_1 * x_2 = [z_1, \dots, z_n]$.

A direct verification shows that this multiplication defines on $\mathbb{C}(X)$ the structure of an algebra. By construction, the obvious embedding

$$\rho_X : X \hookrightarrow \mathbb{C}(X)$$

is a representation. Similarly to the case of ordinary groups, any representation $\rho : X \rightarrow L$ of an n -valued group to a \mathbb{C} -algebra L can be extended to an algebra homomorphism $\mathbb{C}(X) \rightarrow L$.

Let $X = G/A$ be the coset group of a pair (G, A) . Then the group algebra $\mathbb{C}(X)$ is isomorphic to the subalgebra of A -invariants in the group algebra $\mathbb{C}(G)$ of the ordinary group G .

Let X be a multivalued group on the set of irreducible unitary representation of a group G (see Sec. 10). Then the group algebra $\mathbb{C}(X)$ is isomorphic to the classical ring $\mathcal{R}(G)$ of unitary representations of G .

In [11], it is shown that the class of group algebras of involutive multivalued groups is isomorphic to the class of so-called combinatorial algebras, which plays an important role in algebraic combinatorics and the theory of algebra duality (see [2]).

21. ALGEBRAIC ACTION, MULTIVALUED DYNAMICS AND REPRESENTATIONS ON GRAPHS.

For a given action

$$\varphi: X \times Y \rightarrow (Y)^n,$$

define Γ_x , the **graph of the action** of an arbitrary element $x \in X$, as the subset of $Y \times Y$, which consists of the pairs (y_1, y_2) such that $y_2 \in \varphi(x, y_1)$.

Definition. The action of an n -valued group X on an algebraic variety M is called **algebraic** if the action of any element of X is determined by an algebraic correspondence, i.e., its graph is an **algebraic subset** in $M \times M$.

Any equation $T(x, y) = 0$, where T is an order n polynomial in y , defines an n -valued map (or a multivalued dynamics) $\mathbb{C} \rightarrow (\mathbb{C})^n$ under which x is taken to the set of roots $[y_1, y_2, \dots, y_n]$ of $T(x, y)$. In the general case, the number of different images of a point grows exponentially with the number of iterations of the map. In exceptional cases, the growth is polynomial. For example, *for any fixed y every polynomial $p_n(z; x, y)$ (see Sec. 5) defines an n -valued dynamic $p_n(\cdot; \cdot, y) : \mathbb{C} \rightarrow (\mathbb{C})^n : x \rightarrow [z_1, \dots, z_n]$, where the growth is polynomial.*

The polynomial (the Euler-Chasles correspondence)

$$T(x, y) = Ax^2y^2 + Bxy(x + y) + C(x^2 + y^2) + Dxy + E(x + y) + F$$

defines the 2-valued dynamics, in which the number of different images after the k -th iteration is $k + 1$ rather than 2^k as one could expect.

It is known that for the Euler-Chasles correspondence $T(x, y) = 0$ there exists an even elliptic function $f(z)$, of degree 2, such that if $x = f(z)$, then $[y_1, y_2] = [f(z + a), f(z - a)]$ for some a .

For each point $a \in E$, where E is an elliptic curve, an action of the group of integers

$$\varphi_a : \mathbb{Z} \times E \longrightarrow E, \quad \varphi_a(k, z) = z + ka$$

(dynamics with discrete time) is well defined. This action commutes with that of the group \mathbb{Z}_2 (inverting the sign: $\varepsilon\varphi_a(k, z) = -z - ka = \varphi_a(\varepsilon(k), \varepsilon(z))$), whence a coset action of the 2-valued group $\mathbb{Z}_+ = \mathbb{Z}/\mathbb{Z}_2$ on the quotient space $E/\mathbb{Z}_2 \simeq \mathbb{C}P^1$ is well defined.

This means that the Euler-Chasles correspondence is the projection of the mapping $z \rightarrow z + a$ of the elliptic curve E into itself to the projective line $\mathbb{C}P^1$ which is the coset group E/\mathbb{Z}_2 , where \mathbb{Z}_2 is acting on E as $z \rightarrow -z$.

Thus, we have the algebraic action on $\mathbb{C}P^1$ of the two-valued group $\mathbb{Z}_+ = \mathbb{Z}/\mathbb{Z}_2$ with the multiplication

$$x * y = [x + y, |x - y|].$$

This yields

Theorem 13. *The Euler–Chasles correspondence defines the 2-valued dynamics*

$$T_a : \mathbb{C}P^1 \longrightarrow (\mathbb{C}P^1)^2 \simeq \mathbb{C}P^2, \quad a \in E,$$

which is integrable by means of the 2-valued group \mathbb{Z}_+ .

The following result is obtained in [10] by A. P. Veselov and the present author:

Theorem 14. *All algebraic actions of the two-valued group \mathbb{Z}_+ on $\mathbb{C}P^1$ are generated either by the Euler–Chasles correspondence or by a reducible correspondence.*

In some cases it is convenient to represent the graph Γ_x of the action $\varphi: X \times Y \rightarrow (Y)^n$ as the graph with the vertices $y \in Y$ and edges connecting the pairs of vertices (y_1, y_2) such that $y_2 \in \varphi(x, y_1)$, i.e., as the graph of the trajectories of the action of an element $x \in X$ on Y .

In papers [30, 31, 32] by P. V. Yagodovskii, the theory of representations of n -valued groups on graphs is developed. This allowed one to establish a relationship of coset and bicoset multivalued groups with the well-known problem of classification of symmetric and distance transitive graphs, as well as close to the latter distance regular and strictly regular graphs.

22. n -HOPF ALGEBRAS AND FROBENIUS n -HOMOMORPHISMS.

Associate to a space X the ring of functions $\mathbb{C}[X]$. If there are no additional restrictions, then one may assume that $\mathbb{C}[X]$ is the ring of all continuous complex valued functions on X .

For any positive integer k , we have the canonical map

$$s_k : \mathbb{C}[X] \longrightarrow \mathbb{C}[(X)^k],$$

such that

$$s_k(f)[x_1, \dots, x_k] = \sum_{i=1}^k f(x_i).$$

Definition. Let X be an n -valued group with the multiplication

$$\mu : X \times X \longrightarrow (X)^n.$$

The *diagonal map*

$$\Delta : \mathbb{C}[X] \longrightarrow \mathbb{C}[X \times X] \approx \mathbb{C}[X] \hat{\otimes} \mathbb{C}[X]$$

is the linear map $\Delta = \frac{1}{n}F$, where

$$F(f)(x, y) = s_n(f)(\mu(x, y)) = \sum_{i=1}^n f(z_i)$$

and $\mu(x, y) = x * y = [z_1, \dots, z_n]$.

It is well known that for $n = 1$ the diagonal map Δ allows one to introduce a Hopf algebra structure on the ring of functions $\mathbb{C}[X]$. The following statement can be deduced directly from the axioms of multivalued groups.

Lemma 11. *The ring of functions $\mathbb{C}[X]$ on an n -valued group X is a coalgebra $(\mathbb{C}[X], \Delta, \varepsilon)$, where Δ is the diagonal map introduced above, and the counit $\varepsilon : \mathbb{C}[X] \rightarrow \mathbb{C}$ is induced by the embedding of the unit $e \rightarrow X$.*

An action $\varphi : X \times Y \rightarrow (Y)^n$ of an n -valued group X on Y defines on the ring of functions $\mathbb{C}[Y]$ a comodule $(\mathbb{C}[Y], \Delta_Y)$ over the coalgebra $(\mathbb{C}[X], \Delta, \varepsilon)$ structure, where

$$\Delta_Y : \mathbb{C}[Y] \longrightarrow \mathbb{C}[X \times Y] \approx \mathbb{C}[X] \widehat{\otimes} \mathbb{C}[Y] : \Delta_Y(g)(x, y) = \frac{1}{n} \sum_{i=1}^n g(y_i),$$

where $[y_1, \dots, y_n] = \varphi(x, y)$.

The $\text{inv} : X \rightarrow X$ map axiom implies that there is a map $\text{inv}^\perp : X \rightarrow (X)^{n-1}$ such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{d} & X \times X & \xrightarrow{1 \times \text{inv}} & X \times X & \xrightarrow{\mu} & (X)^n \\ & \searrow & & & \searrow & \nearrow & \\ & & & & (X)^{n-1} & & \end{array}$$

inv^\perp (arrow from X to $(X)^{n-1}$), i_n (arrow from $(X)^{n-1}$ to $(X)^n$)

is commutative; here $i_n[x_1, \dots, x_{n-1}] = [x_1, \dots, x_{n-1}, e]$.

This diagram states that the homomorphism $s_n(\cdot)(\mu(x, \text{inv } x)) : \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ is split into composition of homomorphisms $(\text{inv}^\perp)^* i_n^* s_n(\cdot)$.

If $n = 1$, then this algebraic condition determines the antipode $(\text{inv})^* : \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ in the Hopf algebra $\mathbb{C}[X]$.

The multiplication and the comultiplication in a Hopf algebra are related by the fact that the diagonal map is an algebra homomorphism. In papers by E. Rees and the author, a new algebraic notion, that of an ring n -homomorphism (see [9]), was introduced and a definition of n -Hopf algebra was given in order to characterize this relation in the case of n -valued groups. In [12, 14, 15], applications of n -Hopf algebras were developed, based on the following generalization of a classical result about Hopf algebras: *If X is a topological n -valued group, then the ring $\mathbb{C}[X]$ of \mathbb{C} -valued functions on X and the cohomology algebra $H^{2*}(X; \mathbb{C})$ are n -Hopf algebras.*

As a consequence, one can prove that the spaces $\mathbb{C}P^m$ for $m > 1$ do not admit the structure of a 2-valued group. Another important corollary of this result was obtained by T. E. Panov in [26]:

The only (up to homotopy equivalence) simply connected 4-dimensional manifolds admitting a structure of 2-Hopf algebra in the cohomology algebra are $k\mathbb{C}P^2 \# (6 - k)(-\mathbb{C}P^2)$ and $3(S^2 \times S^2)$. Here $\#$ denotes the connected sum of manifolds, the "-" sign before $\mathbb{C}P^2$ means the inversion of orientation, and lM denotes the connected sum of l copies of a manifold M .

Below we give a simpler, although equivalent to the one above, definition of n -Hopf algebra, using the notion of Frobenius n -homomorphism introduced and studied in a series of papers by E. Rees and the author [13, 16, 18].

This notion concerns linear maps of an algebra A with unit to a commutative algebra B . Consider so-called trace homomorphisms $f : A \rightarrow B$, that is, homomorphisms such that $f(a_1a_2) = f(a_2a_1)$. For a given f , define the polylinear homomorphisms

$$\Phi_k(f) : \underbrace{A \otimes \cdots \otimes A}_k \rightarrow B$$

(here we make use of tensor products over \mathbb{C} of \mathbb{C} -modules) by induction in the following way:

$$\begin{aligned} \Phi_1(f) &= f, \\ \Phi_2(f)(a_1, a_2) &= f(a_1)f(a_2) - f(a_1a_2), \\ &\dots \quad \dots \\ \Phi_{k+1}(f)(a_1, \dots, a_{k+1}) &= f(a_1)\Phi_k(f)(a_2, \dots, a_{k+1}) - \sum_{l=2}^{k+1} \Phi_k(f)(a_2, \dots, a_l a_l, \dots, a_{k+1}). \end{aligned}$$

Definition. A linear homomorphism $f : A \rightarrow B$ is called an **Frobenius n -homomorphism** if

- (1) $f(1) = n$;
- (2) $\Phi_{n+1}(f) \equiv 0$.

It is clear that a 1-homomorphism is an ordinary ring homomorphism.

Example. Consider the map

$$s_2 : \mathbb{C}[X] \longrightarrow \mathbb{C}[(X)^2], \quad s_2(f)([x_1, x_2]) = f(x_1) + f(x_2).$$

Note that $s_2(1) = 2$. Now,

$$\begin{aligned} \Phi_2(s_2) : \mathbb{C}[X] \otimes \mathbb{C}[X] &\longrightarrow \mathbb{C}[(X)^2], \\ \Phi_2(s_2)(f_1, f_2) &= s_2(f_1)s_2(f_2) - s_2(f_1f_2). \end{aligned}$$

We have

$$\begin{aligned} \Phi_2(s_2)(f_1, f_2)[x_1, x_2] &= \\ &= (f_1(x_1) + f_1(x_2))(f_2(x_1) + f_2(x_2)) - (f_1(x_1)f_2(x_1) + f_1(x_2)f_2(x_2)) = \\ &= f_1(x_1)f_2(x_2) + f_1(x_2)f_2(x_1) \end{aligned}$$

$$\begin{aligned} \Phi_3(s_2) : \mathbb{C}[X] \otimes \mathbb{C}[X] \otimes \mathbb{C}[X] &\longrightarrow \mathbb{C}[(X)^2], \\ \Phi_3(s_2)(f_1, f_2, f_3) &= s_2(f_1)\Phi_2(s_2)(f_2, f_3) - \Phi_2(s_2)(f_1f_2, f_3) - \Phi_2(s_2)(f_2, f_1f_3). \end{aligned}$$

Hence

$$\begin{aligned} \Phi_3(s_2)(f_1, f_2, f_3)[x_1, x_2] &= (f_1(x_1) + f_1(x_2))(f_2(x_1)f_3(x_2) + f_2(x_2)f_3(x_1)) - \\ &- (f_1(x_1)f_2(x_1)f_3(x_2) + f_1(x_2)f_2(x_2)f_3(x_1)) - (f_2(x_1)f_1(x_2)f_3(x_2) + f_2(x_2)f_1(x_1)f_3(x_1))) = 0 \end{aligned}$$

Thus, s_2 is a 2-homomorphism.

This definition yields a recurrent description of n -homomorphisms. In [16], an explicit formula for n -homomorphisms is given. Using this formula, we obtain the following important result.

Lemma 12. *The linear homomorphism*

$$s_n : \mathbb{C}[X] \longrightarrow \mathbb{C}[(X)^n] : s_n(f)[x_1, \dots, x_n] = f(x_1) + \dots + f(x_n)$$

is an n -homomorphism.

Below we shall require the following properties of Frobenius homomorphisms:

1. Consider the group of linear homomorphisms $\text{Hom}_{\mathbb{C}}(A, B)$. Let $f_i \in \text{Hom}_{\mathbb{C}}(A, B)$, $i = 1, 2$. If f_i is an n_i -homomorphism, then $f_1 + f_2$ is an $(n_1 + n_2)$ -homomorphism.
2. Let $f_1 : A_1 \rightarrow A_2$ be an n -homomorphism and let $f_2 : A_2 \rightarrow B$ be an m -homomorphism. Then the composition

$$f_2 f_1 : A_1 \longrightarrow A_2 \longrightarrow B$$

is an $(n + m)$ -homomorphism.

For each n -valued group X , we construct a linear homomorphism

$$F : \mathbb{C}[X] \longrightarrow \mathbb{C}[X] \widehat{\otimes} \mathbb{C}[X],$$

which is a composition of an n -homomorphism s_n and a ring homomorphism induced by the map μ . Therefore, F is an n -homomorphism.

Definition. Let A be a commutative algebra over a ring k containing $\frac{1}{n}$. We say that A is endowed with an n -bialgebra structure (A, Δ, ε) if

1. $\varepsilon : A \rightarrow k$ is a ring homomorphism;
2. $\Delta : A \rightarrow A \otimes A$ is a linear homomorphism making A a coalgebra (A, Δ, ε) , i.e., the coassociativity axioms for Δ and counit axioms for ε are satisfied;
3. the linear homomorphism $F = n\Delta : A \rightarrow A \otimes A$ is a Frobenius n -homomorphism.

Definition. An n -bialgebra (A, Δ, ε) is an n -Hopf algebra $(A, \Delta, \varepsilon, \chi, \chi^\perp)$ if

1. $\chi : A \rightarrow A$ is a ring homomorphism;
2. $\chi^\perp : A \rightarrow A$ is a Frobenius $(n - 1)$ -homomorphism such that the Frobenius n -homomorphism

$$A \xrightarrow{F} A \otimes A \xrightarrow{1 \otimes \chi} A \otimes A \xrightarrow{m} A$$

splits into a sum $(\eta\varepsilon) + \chi^\perp$, where m is the multiplication in A and $\eta : k \rightarrow A$ is the structure map defining $1 \in A$.

Theorem 15. *Let X be an n -valued group. Then the ring $\mathbb{C}[X]$ carries a structure of an n -Hopf algebra $(\mathbb{C}[X], \Delta, \varepsilon, \chi, \chi^\perp)$, where Δ is the diagonal homomorphism introduced above, ε, χ are ring homomorphisms induced by the maps $e \rightarrow X$ and $\text{inv} : X \rightarrow X$, respectively, and χ^\perp is the $(n - 1)$ -homomorphism $(\text{inv}^\perp)^* s_{n-1}$.*

For a given space X , denote by $\Phi_n(\mathbb{C}[X], \mathbb{C})$ the space of all n -homomorphisms of the ring $\mathbb{C}[X]$ to \mathbb{C} . In paper [16] by E. Rees and the author, the following theorem is proved:

If X is a compact Hausdorff space and the function space $\mathbb{C}[X]$ has the supremum norm, then the map

$$\mathcal{E} : (X)^n \longrightarrow \Phi_n(\mathbb{C}[X], \mathbb{C}) : \mathcal{E}([x_1, \dots, x_n])(f) = s_n(f)([x_1, \dots, x_n]) = f(x_1) + \dots + f(x_n)$$

is a homeomorphism when the space of continuous linear functionals on $\mathbb{C}[X]$ has the weak topology.

Using this theorem, we obtain the following result:

Theorem 16. *If X is a compact Hausdorff space and the ring of functions $\mathbb{C}[X]$ is endowed with an n -Hopf algebra structure $(\mathbb{C}[X], \Delta, \varepsilon, \chi, \chi^\perp)$, then X is an n -valued group.*

Доказательство. By the definition of n -Hopf algebra, the diagonal homomorphism Δ determines an n -homomorphism

$$F : \mathbb{C}[X] \longrightarrow \mathbb{C}[X \times X].$$

Now using the fact that the composition of the n -homomorphism F with an arbitrary ring homomorphism $\mathbb{C}[X \times X] \rightarrow \mathbb{C}$ is an n -homomorphism, we obtain the induced map

$$F^* : \Phi_1(\mathbb{C}[X \times X], \mathbb{C}) \longrightarrow \Phi_n(\mathbb{C}[X], \mathbb{C}).$$

Applying the theorem by E. Rees and the author stated above, one can identify

$$\begin{aligned} \Phi_1(\mathbb{C}[X \times X], \mathbb{C}) &\approx X \times X, \\ \Phi_n(\mathbb{C}[X], \mathbb{C}) &\approx (X)^n. \end{aligned}$$

Hence F^* determines a map

$$\mu : X \times X \longrightarrow (X)^n.$$

The ring homomorphism $\varepsilon : \mathbb{C}[X] \rightarrow \mathbb{C}$ determines the point $e \in X$, and the ring homomorphism $\chi : \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ determines the map $\text{inv} : X \rightarrow X$.

A direct verification shows that the n -Hopf algebra structure $(\mathbb{C}[X], \Delta, \varepsilon, \chi, \chi^\perp)$ yields an n -valued group structure $\mathcal{X} = (X, \mu, e, \text{inv})$ on X . \square

Note that the proof of Theorem 16 in the case where X is a finite set was obtained in [14].

СПИСОК ЛИТЕРАТУРЫ

- [1] J. F. Adams, *Vector fields on spheres.*, Ann. Math., 75, N 3, 1962, 603–632.
- [2] E. Bannai, T. Ito, *Algebraic combinatorics.*, The Benjamin/Cummings PC, 1984.
- [3] V. M. Buchstaber, S. P. Novikov, *Formal groups, power systems and Adams operators.*, Russian Mat. Sb. (N.S.) 84(126), 1971, 81–118.
- [4] V. M. Buchstaber, *Two-valued formal groups. Some applications to cobordism.*, Uspehi Mat. Nauk, 26, no 3, 1971, 195–196.
- [5] V. M. Buchstaber, *Classification of two-valued formal groups.*, Uspehi Mat. Nauk, 28, no 3, 1973, 173–174.
- [6] V. M. Buchstaber, *Characteristic cobordism classes and topological applications of the theories of one-valued and two-valued formal groups.*, J. Sov. Math. 11, 1979, 815–921.

- [7] V. M. Buchstaber, A. N. Kholodov, *Topological constructions connected with multivalued formal groups.*, Math. USSR-Izv., 46, no. 1, 1982, 3–27.
- [8] V. M. Buchstaber, *Functional equations that are associated with addition theorems for elliptic functions, and two-valued algebraic groups.*, Russian Math. Surveys 45, no. 3, 1990, 213–215.
- [9] V. M. Buchstaber, E. G. Rees, *Multivalued groups and Hopf n -algebras.*, Russian Math. Surveys, 51, no. 4, 1996, 727–729.
- [10] V. M. Buchstaber, A. P. Veselov, *Integrable correspondences and algebraic representations of multivalued groups.*, Internat. Math. Res. Notices, no. 8, 1996, 381–400.
- [11] V. M. Buchstaber, A. M. Vershik, S. A. Evdokimov, I. N. Ponomarenko, *Combinatorial algebras and multivalued involutive groups.*, Funct. Anal. Appl., 30, no. 3, 1996, 158–162.
- [12] V. M. Buchstaber, E. G. Rees, *Multivalued groups, their representations and Hopf algebras.*, Transform. Groups, 2, no. 4, 1997, 325–349.
- [13] V. M. Buchstaber, E. G. Rees, *Frobenius k -characters and n -ring homomorphisms.*, Russian Math. Survey 52, no. 2, 1997, 159–160.
- [14] V. M. Buchstaber, E. G. Rees, *Converse theorem.*, www.maths.ed.uk/ elmer, March 1997.
- [15] V. M. Buchstaber, E. G. Rees, *Multivalued groups, n -Hopf algebras and n -ring homomorphisms.*, Lie groups and Lie algebras, Math. Appl., 433, Kluwer, 1998, 85–107.
- [16] V. M. Buchstaber, E. G. Rees, *The Gelfand map and symmetric products.*, Selecta Math. (N.S.), 8, no. 4, 2002, 523–535.
- [17] V. M. Buchstaber, M. I. Monastyrsky, *Generalized Kramers-Wannier duality for spin systems with non-commutative symmetry.*, J. Phys. A, 36, no. 28, 2003, 7679–7692.
- [18] V. M. Buchstaber, E. G. Rees, *Rings of continuous functions, symmetric products, and Frobenius algebras.*, Russian Math. Survey 59, no. 1, 2004, 125–145.
- [19] V. M. Buchstaber, D. V. Leykin, *Addition laws on Jacobian varieties of plane algebraic curves.*, Nonlinear Dynamics, Collected Papers, Proceedings of the Steklov Math. Inst. v. 251, 2005, 49–120.
- [20] V. M. Buchstaber, I. M. Krichever, *Integrable equations, addition theorems and Riemann-Schottky problem.*, Russian Math. Survey 61, no. 1, 2006, 25–84.
- [21] C. W. Curtis, I. Reiner, *Representation theory of finite groups and associative algebras.*, John Wiley & Sons, New York, London, 1962.
- [22] J. Delsarte, *Hypergroupes et operatures de permutation et de transformation.*, Colloque international du CNRS, 71, 1956, 29–45.
- [23] A. N. Kholodov, *The algebraic theory of multivalued formal groups.*, Math. USSR - Sb., 114, no. 2, 1981, 299–321.
- [24] A. N. Kholodov, *Multidimensional two-valued commutative formal groups.*, Russian Math. Surveys, 43, no. 1, 1988, 243–244.
- [25] G. L. Litvinov, *Hypergroups and hypergroup algebras.*, J. Soviet Math., 38, 1987, 1734–1761.
- [26] T. E. Panov, *On the structure of the 2-Hopf algebra in the cohomology of four-dimensional manifolds.*, Russian Math. Surveys 51, no. 1, 1996, 155–157.
- [27] A. P. Veselov, *Growth and integrability in dynamics of mappings.*, Commun. Math. Phys., 145, 1992, 181–193.
- [28] P. V. Yagodovskiĭ, *Deformations of multivalued groups.*, Russian Math. Surveys, 52, no. 3, 1997, 623–624.
- [29] P. V. Yagodovskiĭ, *Linear deformation of discrete groups, and constructions of multivalued groups.*, Izv. Math., 64, no. 5, 2000, 1065–1089.
- [30] P. V. Yagodovskiĭ, *Representations of multivalued groups on graphs.*, Russian Math. Surveys, 57, no. 1, 2002, 173–174.
- [31] P. V. Yagodovskiĭ, *Homogeneous deformations of discrete groups.*, J. Math. Sci. (N. Y.), 113, no. 5, 2003, 728–730.
- [32] P. V. Yagodovskiĭ, *Bicoset groups and symmetric graphs.*, J. Math. Sci. (N. Y.), 126, no. 2, 2005, 1133–1139.

n -значные группы: теория и приложения.

АБСТРАКТ.

Дан обзор наиболее важных результатов теории n -значных групп и некоторых ее приложений. Обсуждаются основные направления исследований. Изложение начинается с исходных определений и разворачивается как последовательность ключевых примеров, опирающихся на результаты из различных областей математики: алгебры, анализа, теории представлений, топологии и теории динамических систем. Содержание ориентировано на широкий круг читателей.

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