Topics in Topology and Mathematical Physics

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Semigroups of Maps into Groups, Operator Doubles, and Complex Cobordisms

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In this paper we develop the theory of Lie groups and Hopf algebras defined in terms of the space $G^V$ of maps taking the space $V$ to the group $G$. The results are based on a construction of a semigroup $G^V_\alpha$ associated to each action $\alpha$ of the group $G$ in $V$. This semigroup is realized as the semigroup of multiplicative operators in the corresponding Novikov operator double. Important examples of semigroups $G^V_\alpha$ have geometric realizations in terms of complex cobordism theory: the semigroup of multiplicative operations in complex cobordisms coincides with the semigroup of maps into itself of the group of diffeomorphisms of the line, the quantum double of the Landweber-Novikov algebra is a subalgebra in the algebra of operations of the doubled theory of complex cobordisms. Part of the main results of this paper appeared in [2].

§1. Semigroups of maps into groups

Suppose $G$ is a group with multiplication $m(g_2, g_1) = g_2 g_1$ and $G'$ is the group with the opposite multiplication $m'(g_2, g_1) = g_1 g_2$. For a space $V$ with a right action $\alpha$ of the group $G$, let us introduce the multiplications $m_\alpha$ and $m'_\alpha$ in the space of maps $G^V$:

1) $m_\alpha : \varphi_2 \ast \varphi_1 (v) = \varphi_2(v) \varphi_1 (v \varphi_2(v))$,
2) $m'_\alpha : \varphi_2 \ast \varphi_1 (v) = \varphi_1 (v \varphi_2(v)^{-1}) \varphi_2(v)$,

where $\varphi_k \in G^V, k = 1, 2, \nu g = \alpha(v, g)$.

In the case when $\alpha$ acts trivially, the corresponding multiplication coincides with the pointwise multiplication in $G^V$ and $(G')^V$.

Denote by $G^V_\alpha$ and $(G^V_\alpha)'$ the spaces with multiplication by $m_\alpha$ and $m'_\alpha$, respectively. Let $i : G \rightarrow G^V$ be the map that takes the point $g$ to the constant map to this point.

**Lemma 1.1.** 1) The spaces $G^V_\alpha$ and $(G^V_\alpha)'$ are semigroups with two-sided units; 2) the inclusion $i$ induces homomorphisms $G \rightarrow G^V_\alpha$ and $G \rightarrow (G^V_\alpha)'$.

**Proof.** Let us check associativity for the multiplication $m_\alpha$:

$(\varphi_3 \ast (\varphi_2 \ast \varphi_1))(v) = \varphi_3(v) (\varphi_2 \ast \varphi_1)(v) = \varphi_3(v)(\varphi_2(v_1)\varphi_1(v_2))$, 1991 Mathematics Subject Classification. Primary 16W30, 57R77.
where \( v_1 = v \varphi_3(v) \) and \( v_2 = v \varphi_2(v_1) \). On the other hand,

\[
((\varphi_3 \ast \varphi_2) \ast \varphi_1)(v) = (\varphi_3 \ast \varphi_2)(v) \varphi_1(v) = (\varphi_3(v) \varphi_2(v_1)) \varphi_1(v_2).
\]

The other statements of the lemma can be verified just as easily. □

The semigroups \( G^V \) possess the following obvious functorial properties.

**Lemma 1.2.** 1) On the space \( V \) with action \( \alpha \) of the group \( G \), any homomorphism \( \lambda: G_1 \to G_2 \) induces the action \( \lambda^* \alpha \) of the group \( G_1 \) and determines a homomorphism

\[
\lambda^*: G^V_1 \to G^V_2.
\]

2) Let \( V_1 \) and \( V_2 \) be spaces with actions \( \alpha_1 \) and \( \alpha_2 \) of the group \( G \), respectively. Then a map \( \gamma: V_1 \to V_2 \) that commutes with the actions \( \alpha_1 \) and \( \alpha_2 \) defines a homomorphism \( \gamma^*: G^V_{\alpha_1} \to G^V_{\alpha_2} \).

For example, the isomorphism \( \lambda: G' \to G, g \mapsto g^{-1} \) induces the isomorphism \( \lambda^*: G^V_{\alpha} \to (G^V_{\alpha})' \) since we have \((G^V_{\alpha})' \simeq (G')^V_{\lambda^*\alpha}\) according to (2).

In the case when the space \( V \) is supplied with a multiplication

\[
\mu: V \times V \to V, \quad \mu(v_1, v_2) = v_1v_2,
\]

the following diagonal map is defined:

\[
(3) \quad \Delta: G^V \to (G \times G)^{V \times V}, \quad (\Delta \varphi)(v_1, v_2) = (\varphi(v_1v_2), \varphi(v_1v_2)).
\]

**Corollary 1.1.** Suppose \( \alpha, \beta, \gamma \) are actions of the group \( G \) on \( V \) compatible with the multiplication \( \mu \) in the sense that

\[
(4) \quad (v_1 \cdot v_2)\alpha(g) = (v_1 \beta(g))(v_2 \gamma(g)).
\]

Then the diagonal defines the homomorphism

\[
(5) \quad \Delta_{\alpha, \beta}: G^V \to (G \times G)^{V \times V}.
\]

**Proof.** The map \( \Delta \) splits into the composition

\[
G^V \xrightarrow{\mu} G^{V \times V} \xrightarrow{\delta} (G \times G)^{V \times V},
\]

where \( \delta: G \times G \to G \) is the usual diagonal. Then, according to Lemma 1.2, the map \( \delta \) induces the homomorphism \( G^V_{\alpha, \beta} \to (G \times G)^{V \times V}_{\delta, (\beta \times \gamma)} \), while the map \( \mu \) induces \( G^V_{\alpha} \to G^V_{\delta, (\beta \times \gamma)} \). □

For each group \( G \) there are three canonical actions of the group on itself: the right translation \( r(g, g_1) = gg_1 \), the left translation \( l(g, g_1) = g^{-1}g_1 \), and the inner conjugation \( ad(g, g_1) = g^{-1}gg_1 \). The semigroups \( G^G, G^L, \) and \( G^G_{ad} \) corresponding to these actions will be the focus of our attention in the sequel.

An important particular case of Corollary 1.1 is the following
Corollary 1.2. The group multiplication $G \times G \to G$ defines the diagonal homomorphisms

$$
\Delta_r : G^r \to (G \times G)^{G \times G}, \quad \Delta_l : G^l \to (G \times G)^{G \times G}, \quad \Delta_{ad} : G_{ad} \to (G \times G)_{ad \times ad}
$$

that satisfy the coassociativity conditions

$$
(\Delta_r \times 1)\Delta_r = (1 \times \Delta_{ad})\Delta_r; \\
(\Delta_{ad} \times 1)\Delta_l = (1 \times \Delta_l)\Delta_l; \\
(\Delta_{ad} \times 1)\Delta_{ad} = (1 \times \Delta_{ad})\Delta_{ad}.
$$

Corollary 1.3. For any space $V$ with a right action $\alpha$ of the group $G$, the diagonal homomorphism $\Delta : G \to G$ can be extended to the homomorphism $\Delta^V : G^V \to (G \times G)^{V \times G}$.

Proof. To the map $f : V \to G$ we assign the map $\Delta(f) : V \times G \xrightarrow{\alpha} V \xrightarrow{f} G \xrightarrow{\Delta} G \times G$.

On $V \times G$ the homomorphism $\Delta$ induces (from the action $\alpha \times \text{ad}$ on $G \times G$) an action of the group $G$. The map $\alpha$ is equivariant with respect to the action of $G$. Therefore we can apply Lemma 1.2. $\square$

Another important example is given by the canonical action of the Lie group $G$ on its Lie algebra $\mathfrak{g}$. In this case we obtain the semigroup $G_{\mathfrak{g} ad}$ with the homomorphism $E^* : G_{\mathfrak{g} ad} \to G_{\mathfrak{g} ad}$ induced by the exponential map $E : g \to G$.

Consider the semigroup of maps $V^V$ of the space $V$ into itself, in which multiplication is the composition of maps. The action $\alpha$ defines a map $j_\alpha : G^V \to V^V$ that takes the map $\varphi : V \to G$ to the map $j_\alpha(\varphi) : V \to V \times G \to V$ given by $j_\alpha(\varphi)(v) = v\varphi(v)$. It is clear that the map $j_\alpha$ is interesting only in the case of nontrivial actions $\alpha$. A direct verification establishes the following

Lemma 1.3. The map $j_\alpha : G^V \to V^V$ is a homomorphism.

To familiarize ourselves with the group $G^V_{\alpha}$, let us consider a few examples.

Example 1.1. Suppose $V$ is the finite set $\{1, \ldots , n\}$ and the action of the group $G$ on $V$ is given by the representation $\rho : G \to S_n$ of the group $G$ into the symmetric group $S_n$. Then $G^V \cong G^n$, and the multiplication is given by the formula

$$
(g_1, \ldots , g_n) \ast (f_1, \ldots , f_n) = (g_1 f_{i_1}, \ldots , g_n f_{i_n}),
$$

where $i_k = \rho(g_k)(k)$, $k = 1, \ldots , n$. In particular, when $V = \{\pm\}$ and $G = O(n)$ is the group of orthogonal matrices, while the action of $G$ on $V$ is given by multiplication by the determinant of the matrix, we obtain an unusual semigroup structure on $O(n) \times O(n)$ with the following multiplication defined in accordance to (6). Namely, if $\varphi, \psi : \{\pm\} \to O(n)$ and $\varphi(k) = A_k, \psi(k) = B_k, A_k, B_k \in O(n)$, $k = \pm 1$, then $(\varphi \ast \psi)(k) = A_k B_1$, where $1_k = k \cdot |A_k|$ and where $|A|$ is the determinant of the matrix $A$. 
EXAMPLE 1.2. Suppose $V = G = \mathbb{R}^n$ and the Euclidean space $\mathbb{R}^n$ as the group of $n$-dimensional vectors acts on itself by ordinary translations. Consider the algebra of linear operators $L(\mathbb{R}^n)$ on $\mathbb{R}^n$ and let us regard $\mathbb{R}^n \times L(\mathbb{R}^n)$ as the set of maps of $\mathbb{R}^n$ into itself of the form $\varphi(v) = w + Av$, where $v, w \in \mathbb{R}^n, A \in L(\mathbb{R}^n)$. According to (1), we have

$$
\varphi_2 \ast \varphi_1(v) = \varphi_2(v) + \varphi_1(v + \varphi_2(v)) = (w_1 + w_2 + A_1 w_2) + (A_1 + A_2 + A_1 A_2) v.
$$

Thus $\mathbb{R}^n \times L(\mathbb{R}^n) \cong \mathbb{R}^{n+n^2}$ is closed with respect to the multiplication (1) and one can see directly that (7) defines the structure of a local group in $\mathbb{R}^{n+n^2}$. In the particular case $n = 1$, let us consider the family of actions $\{\alpha_h: \alpha_h(x)(y) = y + hx\}$ of translations along the line $\mathbb{R}^1$. Then, according to (7), for the plane $\mathbb{R}^2$ we obtain:

$$(x_2, y_2) \ast (x_1, y_1) = (x_1 + x_2 + h y_1 x_2, y_1 + y_2 + h y_1 y_2),$$

i.e., the deformation of the ordinary addition ($h = 0$) into the noncommutative one ($h > 0$).

The following construction is related to a Lie group $G$ and its Lie algebra $\mathfrak{g}$. The exponential map $E: \mathfrak{g} \to G$ defines the inclusion

$$(g, v) \mapsto \tau(g, v) = g \exp v,$$

where $v \in \mathfrak{g}$ and $\xi: \mathfrak{g} \to G$. Using the Campbell–Hausdorff formula, we see that the semigroup structure of $G^V$ induces (by means of $\tau$) a semigroup structure in $G \times \mathfrak{g}^V$.

**Lemma 1.4.** For a smooth action $\alpha$ of Lie group $G$ on the manifold $V$ and for the space of smooth maps $G^V$, the inclusion $\tau$ (see (8)) defines a group structure in $G \times \mathfrak{g}^V$ with the multiplication $(g_2, \xi_2) \circ (g_1, \xi_1) = (g_2 g_1, \xi)$, where

$$\xi(v) = g_1^{-1} \xi_2(v) g_1 + \xi_1(v g_2).$$

**Proof.** Consider smooth maps $\varphi_1, \varphi_2 \in G^V$ such that $\varphi_k(v) = g_k \exp h \xi_k(v)$. For their product in $G^V$, according to (1), we have the formula

$$\varphi_2 \ast \varphi_1(v) = (g_2 \exp h \xi_2(v_1)) g_1 \exp h \xi_1(v + \varphi_2(v)) \approx g_2 g_1 \exp h [g_1^{-1} \xi_2(v) g_1 + \xi_1(v g_2) + o(h)].$$

This formula implies the statement of the lemma. □

Suppose $V$ is a linear space and the action $\alpha$ is given by the representation $\rho: G \to L(V)$ of the group $G$ in the algebra of linear operators on $V$. Denote by $L(V, g)$ the space of linear operators from $V$ to $g$. Lemma 1.4 directly implies the following

**Corollary 1.3.** The inclusion $G \times L(V, g) \to G \times \mathfrak{g}^V$ induces on $G \times L(V, g)$ a group structure with the multiplication

$$(g_2, A_2) \circ (g_1, A_1) = (g_2 g_1, g_1^{-1}(A_2 + A_1) g_1 + A_1 (v \cdot \rho(g_2)) A_k \in L(V, g).$$

To conclude this section, let us present one more general statement about the semigroups $G^V$. 
LEMMA 1.5. Any linear representation $\rho: G \to GL(W)$ of the group $G$ in the linear space $W$ (we can then regard $W$ as a left $G$-module) can be extended to a linear representation $\rho_a: G^+_a \to GL(W^+_a)$ of the semigroup $G^+_a$ in the linear space $W^+_a$ in the following way: Suppose $\varphi: V \to G$ and $f: V \to W$; then

$$\rho_a(\varphi)(f)(v) = \rho_a(\varphi(v))f(\varphi(v)).$$

In the case of trivial action $a$, this lemma is the basis for constructing important representations of the group $G^+$. For nontrivial actions $a$ another particular case is also important, namely the case when $W$ is a trivial $G$-module.

§2. Operator doubles of Hopf algebras

Consider a Hopf algebra $X$ over $k$ with unit $1 \in k$, antipode $\gamma: X \to X$, multiplication $m: X \otimes X \to X$, comultiplication $\Delta: X \to X \otimes X$, and augmentation $\varepsilon: X \to k$. Here $k$ is a field, but most of the constructions below work when $k$ is a commutative ring. All tensor products, unless otherwise specified, are over $k$.

DEFINITION 2.1 ([8]). A Milnor module $M$ over the Hopf algebra $X$ is an algebra with unit $1 \in k$ which is also a module over $X$ satisfying

$$x(uv) = \sum x_n'(u)x_n''(v), \quad x \in X, \quad u, v \in M, \quad A x = \sum x_n' \otimes x_n''.$$

DEFINITION 2.2 ([8]). The operator double $MX$ (O-double) of a Milnor module $M$ over the Hopf algebra $X$ is an algebra such that

1) the algebras $M$ and $X$ are embedded in $MX$ as subalgebras and the $\alpha$-linear map $M \otimes X \to MX, \ u \otimes x \to ux$ induced by these embeddings is an additive isomorphism;

2) the following commutation rule holds in $MX$:

$$xu = \sum x_n'(u)x_n''(v), \quad x \in X, \quad u \in M, \quad A x = \sum x_n' \otimes x_n''.$$

When the action of $X$ on $M$ is trivial, we obviously have $MX \cong M \otimes X$.

In the cases when it will be necessary to stress the dependence of the O-double $MX$ on the action of $X$ on $M$, we shall use the notation $M_a X$.

Consider the Hopf algebra $X^* = \Hom_k(X', k)$ dual to the Hopf algebra $X$. Denote by $r, l,$ and $\gamma$ the actions of $X$ on $X^*$ defined by the formulas

$$(r(x)\sigma, y) = \langle \sigma, yx \rangle;$$

$$(l(x)\sigma, y) = \langle \sigma, \gamma(x)y \rangle;$$

$$(\gamma(x)\sigma, y) = \langle \sigma, \sum \gamma(x_n')y x_n'' \rangle,$$

where $x, y \in X, \sigma \in X^*$ and $\langle \cdot, \cdot \rangle$ is the canonical pairing $X^* \otimes X \to k$.

LEMMA 2.1. With respect to the actions $r, l, \gamma$ the O-double $X^* \otimes X$ is a Milnor module over $X$ and therefore the O-doubles $X^*_a X, X^*_a X, X^*_a X$ are defined.

Denote by $f_a$ the ring homomorphism $X \to L(M)$ induced by the action $a: M \otimes X \to M$ of the Hopf algebra $X$ on $M$. The algebra $M_a X$ described in Definition 2.2 was called the operator double by S. P. Novikov, because Definition 2.1 immediately implies the following statement.
LEMMA 2.2. The map \( j_M: M \to L(M), \quad j_M(u)(x) = j_M(u)x \) is a homomorphism of rings.

DEFINITION 2.3 ([8]). The semigroup of multiplicative elements \( m(M_aX) \) of the operator double \( M_aX \) is the multiplicative semigroup of elements \( y \in M_aX \) such that the maps \( j_M(y): M \to M \) are ring homomorphisms.

The paper [8] gives some motivation, using the example of cohomology operations in the theory of complex cobordisms, of the importance of describing such semigroups. The results presented below solve the corresponding problem in the case of O-doubles defined by an action \( \alpha \) of the group \( G \) on the space \( V \).

Suppose \( X^* \) is the Hopf algebra of functions on \( G \) with values in \( k \), while \( X \) is the dual Hopf algebra. As usual in the construction of doubles ([4], [8], [10]), we assume that a basis \( \{e_k\} \) has been chosen in \( X^* \) and \( \{e^k\} \) is the dual basis in \( X \). Then we can write the identity operator from \( X^* \) to \( X^* \) in the form \( \Delta = \sum e_k e^k \in X^*X \) and expand any function \( f \in X^* \) as \( f(\cdot) = \sum e_k e^k \cdot f(\cdot) \).

In the role of \( M \), let us take the ring of \( \alpha \)-regular functions on the space \( V \), i.e., functions such that, given any action \( \alpha \) and any function \( p(v) \in M \), the function \( p(vg) \) belongs to \( M \) for a fixed \( g \) and belongs to \( X^* \) for a fixed \( v \). Then the following action is defined \( a: M \otimes X \to M, \quad a(p(v), x) = \langle x, p(vg) \rangle \in M \).

LEMMA 2.3. The given action \( a \) defines a Milnor module structure on \( M \) over the Hopf algebra \( X \) and therefore the O-double \( M_aX \) is defined.

PROOF. Suppose \( p_1(v), p_2(v) \in M \). For a fixed \( v \) we have the product of two functions \( p_1(v) \cdot p_2(v) \) on the group \( G \) and therefore

\[
\langle x, p_1(vg) \cdot p_2(vg) \rangle = \langle \Delta x, p_1(vg) \otimes p_2(vg) \rangle,
\]

since the comultiplication \( \Delta \) on \( X \) and the multiplication in \( X^* \) are adjoint with respect to the scalar product. \( \Box \)

A direct consequence of the previous results is the following

Theorem 2.1. The map \( j^0_\alpha: G^1_\alpha \to M_aX, \quad j^0_\alpha(\varphi) = \sum e_k(\varphi) e^k \) induces the homomorphism \( G^1_\alpha \to m(M_aX) \) defined as follows: for each \( \varphi \in G^1_\alpha \) the element \( j^0_\alpha(\varphi) \) is the ring homomorphism of the ring of functions \( M \) induced by the map \( j^0_\alpha(\varphi) : V \to V \).

Let us return again to the general case of O-doubles. A straightforward verification establishes the following

LEMMA 2.4. 1) Let \( M_aX \) be some O-double. Then the homomorphism of Hopf algebras \( \lambda: Y \to X \) induces a Milnor action \( \lambda \) on \( M \) and determines a ring homomorphism of the O-doubles \( \lambda_a: M_{\lambda_a} \to M\).

2) Let \( M \) and \( N \) be Milnor modules over the Hopf algebra \( X \) with actions \( a \) and \( b \) on \( X \) respectively. Then any ring homomorphism \( h: M \to N \) equivariant with respect to these actions can be extended to a ring homomorphism of the O-doubles \( h_\alpha: M_aX \to N_\alpha X \).

The restriction of the homomorphism \( j^0_\alpha \) to the subgroup \( G \subset G^1_\alpha \) coincides with the canonical embedding \( j: G \to X \). Lemmas 1.2 and 2.4 show that well-known universality properties of the homomorphism \( j \) can be carried over under certain conditions to the homomorphism \( j^0_\alpha \).
§3. Quantum doubles of Hopf algebras

Suppose that as above $X$ and $X^*$ are dual Hopf algebras over $k$. Fix an additive basis $\{e^k\}$ in $X$ and its dual basis $\{e^k\}$ in $X^*$. Using these bases, let us express the operations in the Hopf algebras:

$$m(e^i \otimes e^j) = e^i e^j = \sum m_{ij}^k e^k; \quad m^*(e_i \otimes e_j) = e_i e_j = \sum \lambda_{ij}^k e^k;$$

$$\Delta e^k = \sum \lambda_{ij}^k e^i \otimes e^j; \quad \Delta^* e^k = \sum m_{ij}^k e_i \otimes e_j;$$

$$\gamma(e^k) = \sum \gamma_i^k e^i; \quad \gamma^*(e^i) = \sum \gamma_i^k e^k.$$

Denote by $X^0$ the Hopf algebra $X$ with the opposite comultiplication, i.e., $\Delta^0 e^k = \sum \lambda_{ij}^k e^i \otimes e^j$.

**Definition 3.1** ([4],[10]). The quantum double of the Hopf algebra $X$ is the Hopf algebra $D(X)$ such that

1. The algebras $X^*$ and $X^0$ are embedded in $D(X)$ as Hopf subalgebras and the $k$-linear map $X^* \otimes X^0 \rightarrow D(X)$, $u \otimes x \rightarrow ux$, induced by these embeddings is a coalgebra isomorphism (here $X^* \otimes X^0$ is regarded as the tensor product of coalgebras over $k$);

2. The commutation rule of elements $u \in X^*$ and $x \in X^0$ in $D(X)$ is described by the equation

$$\gamma^1 \Delta(a) = (\sigma \circ \Delta)(a) \gamma^1,$$

where $\Delta$ is the comultiplication in $D$, the operator $\sigma$ is the transposition in $D(X) \otimes D(X)$, i.e., $\sigma(a \otimes b) = b \otimes a$, while $\gamma$ is the image of the canonical element $\sum e_k \otimes e^k \in X^* \otimes X^0$ under the embedding $X^* \otimes X^0 \subset D(X) \otimes D(X)$.

In order to avoid cumbersome formulas, we shall identify the elements from $X^*$ and $X^0$ with their images in $D(X)$.

Now in $D(X) \otimes D(X) \otimes D(X)$ let us put

$$\gamma_{13} = \sum e_i \otimes 1 \otimes e^s, \quad \gamma_{12} = \sum e_i \otimes e^s \otimes 1, \quad \gamma_{23} = \sum 1 \otimes e_i \otimes e^s.$$

An important consequence of the construction of the quantum double $D(X)$ is the following

**Theorem 3.1** ([4]). The element $\gamma$ is invertible in the algebra $D(X) \otimes D(X)$ and gives a solution to the quantum Yang–Baxter equation (QYBE):

$$\gamma_{12} \gamma_{13} \gamma_{23} = \gamma_{23} \gamma_{13} \gamma_{12};$$

where $\Delta$ is the comultiplication in $D$, the operator $\sigma$ is the transposition in $D(X) \otimes D(X)$, i.e., $\sigma(a \otimes b) = b \otimes a$, while $\gamma$ is the image of the canonical element $\sum e_k \otimes e^k \in X^* \otimes X$ under the embedding $X^* \otimes X \subset D(X) \otimes D(X)$.

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THEOREM 3.2. ([8]) Suppose \( X \) is a cocommutative Hopf algebra. Then the quantum double \( D(X) \) coincides with the O-double \( X^\text{ad} \).

PROOF. By assumption \( X^0 = X \) and \( X^* \) is a commutative algebra. Therefore the operators \( \mathfrak{R}_{12} \) and \( \mathfrak{R}_{13} \) commute. Hence, by Theorem 3.1, we have the relation

\[
\mathfrak{R}_{12} \mathfrak{R}_{13} = \mathfrak{R}_{13} \mathfrak{R}_{12}.
\]

Let us express the left- and right-hand side of equation (15) in the canonical basis of the algebra \( D(X) \otimes D(X) \):

\[
\begin{align*}
\mathfrak{R}_{12} \mathfrak{R}_{13} &= \sum e_\ell \otimes e^i e_j \otimes e^j, \\
\mathfrak{R}_{13}^{-1} \mathfrak{R}_{12} &= \sum \gamma^i(e_\ell) e_j e_i \otimes e^i e^j \\
&= \sum e_\ell e_\ell e_i e_j e_i \otimes e^j e^i \\
&= \sum e_\ell \otimes a^e_i e_i \otimes e^j \\
\end{align*}
\]

where

\[
a^e_i = \left( \sum \lambda_{ikj}^l \gamma^i e^k e^l, e^j \right).
\]

On the other hand, the commutation rule in \( X^\text{ad} \) is of the form (see [8])

\[
\begin{align*}
e^i e_j &= \sum \lambda_{ikj}^l \gamma^i e^k e^l, \\
p_{11} p_{2k}(e_j) &= \sum b^e_i e_i, \\
b^e_i &= \left( \gamma^i e^i, p_{11} p_{2k}(e_j) \right) = \left( \gamma^i e^i e^j, e_j \right).
\end{align*}
\]

Since \( \lambda_{ikj}^l = \lambda_{ikj}^l \), the two commutation rules coincide. \( \Box \)

\[\text{§4. The Hopf comodule structure of O-doubles}\]

We begin with a natural dualization of the Milnor module over a Hopf algebra.

DEFINITION 4.1. By a Hopf comodule (or Milnor comodule) over a Hopf algebra \( X \) we mean a \( k \)-algebra \( M \) with unit provided \( M \) is a comodule over \( X \) with coaction \( \rho : M \rightarrow X \otimes M \) such that \( \rho(uv) = \rho(u) \rho(v) \), i.e., such that \( \rho \) is a homomorphism of rings.

Suppose \( M \) is a Milnor module over the Hopf algebra \( X \) and \( a : X \otimes M \rightarrow M, a(x, u) = x(u) \) is the homomorphism that determines this structure. Then the following homomorphism is defined

\[
b : M \rightarrow X^* \otimes M, \quad b(u) = \sum e_\ell \otimes e^i(u).
\]

Clearly, (16) is the composition of the homomorphism \( a^* : M \rightarrow \text{Hom}(X, M) \) with the canonical isomorphism \( \text{Hom}(X, M) \cong X^* \otimes M \).

A direct verification establishes the following...
LEMMA 4.1. The Milnor module over the Hopf algebra $X$ is a Milnor comodule with the coaction (16) over the dual Hopf algebra $X^*$.

COROLLARY 4.1. For any Milnor module $M$ over a cocommutative Hopf algebra $X$ with left action $a$ on $M$, the diagonal homomorphism $\Delta: X \rightarrow X \otimes X$ extends to a ring homomorphism $\Delta: M_a \rightarrow X_{ad}^* \otimes M_a X$.

PROOF. The algebra $X^* \otimes M$ is a Milnor module over the Hopf algebra $X$ with action $ad \otimes a$. For a cocommutative algebra $X$, the diagonal homomorphism $\Delta: X \rightarrow X \otimes X$ is a homomorphism of Hopf algebras and therefore, according to Lemma 2.4, the following ring homomorphism is defined:

$$(X^* \otimes M)_\Delta X \rightarrow (X^* \otimes M)_{(ad \otimes a)}(X \otimes X) \cong X_{ad}^* X \otimes M_a X.$$ 

On the other hand, the homomorphism $b: M \rightarrow X^* \otimes M$, being dual to the action $a$, is a homomorphism of Milnor $X$-modules. Therefore the homomorphism $M_a X \rightarrow (X^* M)_\Delta X$ is defined. 

Now using Theorem 3.2 and Corollary 4.1, we obtain

THEOREM 4.1. The operator double $M_a X$ of a cocommutative Hopf algebra $X$ is a Hopf comodule over the quantum double $D(X)$.

§5. Some information from the theory of complex cobordisms

A systematic exposition of the foundations of complex cobordism theory can be found, for example, in [11].

Let $U(n)$ be the unitary group of linear transformations of complex linear space $\mathbb{C}^n$. Denote by $i_k$ the standard embedding of unitary groups $i_k: U(n) \subset U(n+1)$.

Consider the universal $n$-dimensional complex vector bundle $\xi \rightarrow BU(n) = \lim_{k \rightarrow \infty} G_n, n+k$, where $G_n, n+k$ is the Grassman manifold of complex $n$-dimensional linear subspaces of $\mathbb{C}^{n+k}$. The embedding $i_k$ induces the map $B_{i_k}^*: BU(n) \rightarrow BU(n+1)$ such that $(B_{i_k}^*)^* \xi_{n+1} = \xi_n \oplus 1$, where 1 denotes the complex trivial linear bundle.

Denote by $M_n = MU(n)$ the Thom space $T(\xi_n) = D(\xi_n)/\partial D(\xi_n)$, where $D(\xi_n)$ and $\partial D(\xi_n)$ are the fiber spaces associated to $\xi_n$ with fiber the unit disk $D^{2n}$ and the sphere $S^{2n-1}$ in $\mathbb{C}^n$, respectively. The point from $M_n$ corresponding to $\partial D(\xi_n)$ is chosen for the base point. An important role is played by the following multiplicativity property of Thom spaces.

Suppose $\xi_k = V_k, k = 1, 2$, are complex vector bundles. Then for the bundle $\xi_1 \times \xi_2 \rightarrow V_1 \times V_2$ we have the homeomorphism $T(\xi_1 \times \xi_2) \cong T(\xi_1) \wedge T(\xi_2)$. Here and below $\wedge$ denotes the smash product of pointed spaces, i.e., $X \wedge Y = X \times Y/X \vee Y$, where $X \vee Y$ is the wedge of pointed spaces. In particular, we have $T(\xi_n \oplus 1) \cong S^2 M_n$, where the sphere $S^2$ is regarded as the Thom space of the trivial complex one-dimensional bundle over the point. Therefore the map $B_{i_k}$ induces the map $M_{i_k}: S^2 M_n \rightarrow M_{n+1}$.

DEFINITION 5.1. The Thom spectrum in the theory of complex cobordisms is the sequence of Thom spaces of universal bundles with the maps that join them:

$$MU = \{M_n = MU(n), M_{i_k}\}.$$
The Thom spectrum $MU$ for pairs of CW-complexes $V \subset W$ is used to define the cobordism groups

$$U^q(W, V) = \lim_{n \to \infty} [S^{2n-q}(W/V), MU(n)]$$

and the bordism groups

$$U_q(W, V) = \lim_{n \to \infty} [S^{2n+q}, (W/V) \wedge MU(n)].$$

Here $[\cdot, \cdot]$ denotes the set of homotopy classes of base point preserving maps.

The point in $W/V$ corresponding to $V$ is chosen to be the base point. We set $U^q(W) = U^q(W, \emptyset)$ and $U_q(W) = U_q(W, \emptyset)$, where $\emptyset$ is the empty set. Recall that by definition $W/\emptyset = W_+$ is the disjoint union of $W$ with the point $*$ regarded as the base point.

One can give a purely geometric description of the groups $U^q(\cdot)$ and $U_q(\cdot)$ by using the notion of bordism of a map of smooth manifolds in the stable tangent (or normal) bundle, provided a complex bundle structure is fixed.

Let us introduce the graded groups

$$U^*(W) = \sum U^q(W), \quad U_*(W) = \sum U_q(W).$$

The standard embedding $U(n) \times U(m) \subset U(n+m)$ induces a map of Thom spaces $\mu_{n, m}: M_n \wedge M_m = M(\xi_n \times \xi_m) \to M_{n+m}$ and thereby a map of Thom spectra

$$\mu: MU \wedge MU \to MU.$$ 

This map allows us to introduce the structure of a graded commutative ring in $U^*(W)$.

According to the Milnor–Novikov theorem, we have the isomorphism

$$U^*(\text{point}) = \Omega U = \mathbb{Z}[a_1, \ldots, a_n, \ldots], \quad \deg a_n = -2n.$$ 

**Definition 5.2.**

1. A Thom $U$-class of a complex bundle $\xi \to V$ is the class of cobordisms $u(\xi) \in U^2(\xi)$ whose representative is the map of Thom spaces $T_\xi \to MU(n)$ corresponding to the classifying map $f: V \to BU(n)$, $f^*\xi_n = \xi$.

2. The Euler class of a complex $n$-dimensional bundle $\xi \to V$ is the class of bordisms $\chi(\xi) = S_0^* u(\xi) \in U^2_n(V)$, where $S_0: V \to T_\xi$ is the embedding corresponding to the zero section of the bundle $\xi$.

Denote by $u(n) \in U^2(MU(n))$ the Thom class of the universal bundle $\xi_n \to BU(n)$. It is clear that the identity map $MU(n) \to MU(n)$ is a representative of the class $u(n)$.

According to Conner and Floyd [11], there exists a complete characteristic Chern class

$$C(\xi) = 1 + C_1(\xi) + \cdots = C_k(\xi) + \cdots, \quad C_k(\xi) \in U^k(V)$$

of the complex vector bundle $\xi \to V$ which is uniquely determined by the following properties:
1) $C_k(\xi) = 0$, if $k > \dim \xi$;
2) $C_k(\xi) = \chi(\xi)$, if $k > \dim \xi$;
3) $C(\xi_1 + \xi_2) = C(\xi_1)C(\xi_2)$.

This result directly implies the following facts:

1) $U^*(\prod CP(\infty)) = \Omega[[u_1, \ldots, u_n]]$, where $u_m = C_1(\eta)$ is the first Chern class of the universal one-dimensional bundle $\xi \to CP(\infty)$ over the infinite-dimensional projective space, regarded here as the $m$th factor of the product space $\prod CP(\infty)$.

2) $U^*(BU(n)) = \Omega[[C_1, \ldots, C_n]]$, where $C_k = C_k(\xi_n)$.

3) The map $h_n: \prod CP(\infty) \to BU(n)$ determined by the embedding of the maximal torus $T_n \subset U(n)$ induces a monomorphism

$$U^*(BU(n)) \to U^*(\prod CP(\infty))$$

under which $C_k$ is taken to the $k$th elementary symmetric polynomial in the variables $u_1, \ldots, u_n$.

4) The embedding $S_n: BU(n) \to BU(MU(n))$ defined by the zero section of the universal bundle $\xi_n$ induces the monomorphism $h^n_* S_n^*: U^*(MU(n)) \to U^*(\prod CP(\infty))$ whose image coincides with the ring of all symmetric formal series in $u_1, \ldots, u_n$ divisible by the monomial $u_1 \cdots u_n = h^n_* S_n^* u(n)$.

Using the construction of complex cobordism theory in terms of the Thom spectrum, we directly obtain the existence of a multiplicative transformation

$$\varepsilon: U^*(W) \to H^*(W; Z),$$

which is entirely determined by the property that for the Thom classes $u(\xi) \in U^{2n}(T\xi)$, the classes $\varepsilon(u(\xi)) \in H^{2n}(T\xi)$ coincide with the ordinary Thom classes in cohomology.

Under the transformation $\varepsilon$, the Chern–Conner–Floyd class $C_k(\xi)$ of the complex vector bundle $\xi \to W$ is taken to the ordinary Chern cohomology classes $C_k^H(\xi)$, i.e., $\varepsilon C_k(\xi) = C_k^H(\xi)$.

Denote by $Z^n_{\infty}$ the semigroup of all $n$-dimensional vectors with nonnegative integer coordinates. Put $Z_{\infty}^{\infty} = \lim_{\to} Z^n_{\infty}$. Suppose $(q) \in Z_{\infty}^{\infty}$ is an infinite-dimensional vector all of whose coordinates except the $q$th is zero, while the $q$th one is equal to 1. Then the semigroup $Z^n_{\infty}$ coincides with the semigroup of all finite linear combinations $\sum n_q(q)$, where $n_q$ are nonnegative integers. For each element $w = \sum n_q(q)$ let us put $\|w\| = \sum n_q(q)$ and $|w| = \sum n_q$. For

$$\omega = \sum_{q=1}^I n_q(q)$$

denote by $\sigma_{\omega}$ the smallest symmetric polynomial in $u_1, \ldots, u_n$, $n \geq |\omega|$, containing the monomial

$$(u_1 \cdots u_{n_1})(u_{n_1+1}^2 \cdots u_{n_1+n_2}^2) \cdots (u_{|\omega|+1}^2 \cdots u_{|\omega|+n_{|\omega|}}^2)$$

and by $C_{\omega}(\xi_n)$ the unique element of the ring $U^{2|\omega|}(BU(n))$, $n > |\omega|$, that satisfies $h^n_* C_{\omega}(\xi_n) = \sigma_{\omega}$. For example, the polynomial $\sigma_{|\omega|}$ is the Newton polynomial $\sum u_i^{q_i}$. 

the polynomial $\sigma_{k(1)}$ is the elementary symmetric polynomial $\sum u_1 \ldots u_k$, while the class $C_{k(1)}(\xi_n)$ is $C_k$.

For each $\omega \in \mathbb{Z}_\infty^+$, let us put

$$s_{\omega, n} = s_{\omega} u(n) = u(n) C_{\omega}(\xi_n) \in U^{2(n+\|\omega\|)}(MU(n)).$$

It follows from the functoriality of Thom classes and of characteristic classes that the sequence of elements $\{s_{\omega, n}|n = 1, 2, \ldots\}$ defines a stable cohomology operation $s_{\omega}$ of degree $2\|\omega\|$ in the theory $U^*(\cdot)$ which acts as follows.

Suppose $a \in U^q(W)$. Choose a representative $f : S^{2n-q} W \to MU(n)$ of the class $a$ and put

$$s_{\omega} a = f^* s_{\omega, n} \in U^{2(n+\|\omega\|)}(S^{2n-q} W) \cong U^{q+2\|\omega\|}(W).$$

Denote by $S = \sum_{n \geq 0} S^n$ the free topological commutative group with topological basis $s_{\omega}, \omega \in \mathbb{Z}_\infty^+$. Let

$$A_U = \sum_{n \geq -\infty} A^n_U$$

be the group of all stable cohomology operations in the theory of complex cobordisms. It is clear that we have the isomorphism

$$A_U = U^*(MU), \quad U^k(MU) = \lim_{n} U^{2n+k}(MU(n)),$$

the inverse limit being induced by the maps $M_i$. With respect to composition, $A_U$ is a graded ring. The operator of multiplication by a scalar $\sigma \in \Omega_U$ is a stable cohomology operation. Therefore we have an embedding of rings $\Omega_U \subset A_U$ and $A_U$ is a module (that we regard as a left module) over $\Omega_U$.

Using the description of the rings $U^*(MU(n))$ given above, we see that $A_U \cong \Omega_U S$, i.e., elements of $A_U$ may be uniquely expressed in the form $\sum_\omega a_\omega s_\omega$. The following results are due to Novikov [7] and Landweber [6]:

1. $S = \sum S^n$ is a subalgebra in $A_U$.
2. The algebra $S$ is a cocommutative Hopf algebra with diagonal

$$\Delta : S \to S \otimes S, \quad \Delta s_\omega = \sum_{\omega_1 + \omega_2 = \omega} s_{\omega_1} \otimes s_{\omega_2}.$$

Here the indices $\omega_1$ and $\omega_2$ are added as elements of the semigroup $\mathbb{Z}_\infty^+$.

3. The multiplication in $S$ can be uniquely determined from the following properties:

(a) for any $n \geq 1$, the ring $\mathbb{Z}[[u_1, \ldots, u_n]]$ is a module over the Hopf algebra $S$ such that $s_{(q)} u_m = u_m^{q+1}$ and $s_{\omega} u_m = 0, \omega \neq (q)$ for all $m = 1, \ldots, n$ and

$$s_{\omega} (ab) = \sum_{\omega_1 + \omega_2 = \omega} s_{\omega_1}(a) s_{\omega_2}(b) \text{ for all } a, b \in \mathbb{Z}[[u_1, \ldots, u_n]].$$

(b) The element $s \in S^n$ vanishes if and only if we have $s(a) = 0$ for all $a \in \mathbb{Z}[[u_1, \ldots, u_n]]$. 
DEFINITION 5.3. The algebra $S$ is called a *Landweber–Novikov algebra*.

It follows directly from the description of the multiplication in the $U^*(\cdot)$ theory and the description of the algebra $S$ that for any CW-complex $W$ the ring $U^*(W)$ is a Milnor module over the Hopf algebra $S$ (see Definition 2.1). The complete description of all the operations in $A_U$ is concluded by the following result due to Novikov [7].

The commutation rule in the algebra $A_U$ is

$$s_{ij} \sigma = \sum_{\omega = \omega_1 + \omega_2} s_{ij}(\sigma)s_{\omega_2}.$$

The operation $\varphi \in A_U$ is called *multiplicative* if it determines a ring homomorphism $\varphi: U^*(V) \to U^*(V)$ for any space $V$. As any other operation, $\varphi$ is determined by the sequence of its values $\{\varphi(u(n)), n = 1, \ldots\}$ on the universal elements $u(n) \in U^*(MU(n))$, but in view of the monomorphism

$$h_n^* S^*: U^*(MU(n)) \to U^*(\prod \mathbb{CP}(\infty)), \quad h_n^* S^* u(n) = u_1 \ldots u_n$$

each multiplicative operation is uniquely determined by its value on the class $u = c_1(\xi_1) \in U^2(\mathbb{CP}(\infty))$, i.e., by the series

$$\varphi u = \varphi(u) = u + \sum_{k \geq 0} \varphi_k u^{k+1}, \quad \varphi_k \in \Omega_U.$$  

Note that the condition $\varphi(u) = u + (u^2)$ follows from the stability of the operation.

Denote by $mA_U$ the set of all multiplicative operations. As we noted above, $mA_U$ may be identified with the set of series of the form (17) from the ring $\Omega_U[[u]]$. It is easy to show that the multiplicative operation $\varphi$ belongs to the Landweber–Novikov algebra $S \subset S^*$ if and only if all the coefficients of the series $\varphi(u)$ belong to the ring of integers $\mathbb{Z} = \Omega^1_U \subset \Omega_U$.

Denote the set of such operations by $mS$. With respect to the composition of operations, $mA_U$ is a semigroup, while $mS$ is a group. The semigroup $mA_U$ and the group $mS$ play the central role in subsequent sections.

Consider the Hopf algebra $S^* = \text{Hom}(S, \mathbb{Z})$ dual to the Landweber–Novikov algebra over the ring of integers $\mathbb{Z}$. Fix an additive basis $\{s_0, \omega \in \mathbb{Z}_+\}$ in $S$ and denote by $s^{(i)}$ the dual basis in $S^*$. It follows immediately from the description of the diagonal $\Delta$ in $S$ that the elements $s^{(q)}$ are primitive in $S$ (i.e., $\Delta s^{(q)} = s^{(q)} \otimes 1 + 1 \otimes s^{(q)}$, $q = 1, 2, \ldots$) and

$$S^* = \mathbb{Z}[s^{(i)}, \ldots, s^{(q)}, \ldots].$$

DEFINITION 5.4. A *Chern–Dold character* in the theory of complex cobordisms is a multiplicative transformation of cohomology theory

$$\text{ch}_U: U^*(W) \to H^*(W; \Omega \otimes \mathbb{Q})$$

uniquely determined by the fact that for $X = (\text{point})$ the homomorphism $\text{ch}_U$ coincides with the canonical embedding $\Omega_U \subset \Omega_U \otimes \mathbb{Q}$. Here $\mathbb{Q}$ is the field of rational numbers.
The theory of the operator \( \text{ch}_L \) was constructed by the author in [1]. It was called the Chern–Dold character because it may be obtained by the Dold construction, which generalizes the well-known Chern character from \( K \)-theory to the case of arbitrary cohomology theories.

The action of the algebra \( A_U \) on the ring \( \Omega_U \otimes \mathbb{Q} \) allows us to regard the rings \( H^*(W; \Omega_U \otimes \mathbb{Q}) \) as \( A_U \)-modules. A most important property of the Chern–Dold character \( \text{ch}_U \) is that it determines an \( A_U \)-module homomorphism.

To the operation \( s_{n_\sigma} \in S \), let us assign the transformation

\[
s_{n_\sigma}^H : U^*(W) \to H^*(W; \mathbb{Z}), \quad s_{n_\sigma}^H(a) = \varepsilon s_{n_\sigma}(a),
\]

where \( \varepsilon : U^*(W) \to H^*(W; \mathbb{Z}) \).

It follows at once from the definitions that for \( W = \text{(point)} \) we obtain a homomorphism \( s_{n_\sigma}^H : \Omega_U \to \mathbb{Z} \) calculating the normal cohomological characteristic numbers, i.e., if the smooth manifold \( M^{2n} \) with complex normal bundle \( v \) represents the class \( \sigma \in \Omega_U^{2n} \), then

\[
s_{n_\sigma}^H(\sigma) = (C_{n_\sigma}^H(v), (M^{2n})),
\]

where \( C_{n_\sigma}^H = \varepsilon C_{n_\sigma} \) is the Chern class and \( (M^{2n}) \) is the fundamental cycle of the manifold \( M^{2n} \) in homology. Let us introduce the ring \( \Omega_U(Z) \) by setting

\[
\Omega_U(Z) = \sum_{n \geq 0} \Omega_U^{2n}(Z),
\]

\[
\Omega_U^{2n}(Z) = \{ \sigma \in \Omega_U^{2n}(Z) \otimes \mathbb{Q}, \quad s_{n_\sigma}^H(\sigma) \in \mathbb{Z} \quad \text{for all } \sigma \}.
\]

It is clear that the ring \( \Omega_U^{2n}(Z) \) is closed with respect to the action of the algebra \( A_U \). Now consider the transformation

\[
\text{ch}_U : U^*(CP(\infty)) = \Omega_U[[u]] \to H^*(CP(\infty); \Omega_U \otimes \mathbb{Q}) = \Omega_U \otimes \mathbb{Q}[[t]],
\]

where \( u \in U^2(CP(\infty)) \) and \( t \in H^2(CP(\infty); \mathbb{Z}) \) are the first Chern classes in complex cobordisms and in the cohomology of the universal bundle \( \xi_1 \to CP(\infty) \).

We have

\[
\text{ch}_U = t + \sum_{k \geq 1} \alpha_k t^{k-1}, \quad \alpha_k \in \Omega_U^{2n} \otimes \mathbb{Q}.
\]

Therefore, for the class \( u(n) \in U^{2n}(MU(n)) \), we obtain

\[
\text{ch} u(n) = t(n) + \sum \alpha_{n} s_{n_\sigma}^H \in H^{2n}(MU(n), \Omega_U \otimes \mathbb{Q}).
\]

Here \( t(n) = \varepsilon u(n), \quad s_{n_\sigma}^H = s_{n_\sigma}^H(u(n)) \) and \( \alpha_{n} = \prod n_{q}(q) \) for \( \omega = \sum n_{q}(q) \). Thus we arrive at the following result ([1]):

1. \( \Omega_U(Z) = \mathbb{Z}[\alpha_1, \ldots, \alpha_q, \ldots] \).

2. The Chern–Dold character \( \text{ch}_U \) can be factored into a composition with the transformation

\[
\text{ch}_U^H : U^*(W) \to H^*(W; \Omega_U(Z))
\]

defined by the following explicit formula

\[
\text{ch}_U^H(a) = \sum \alpha_{n} s_{n_\sigma}^H(a).
\]

In particular, for \( W = \text{(point)} \), the transformation \( \text{ch}_U^H \) defines the canonical representation of the cobordism class \( \sigma \in \Omega_U^{2n} \) in the form of a homogeneous polynomial in the graded variables \( \alpha_1, \ldots, \alpha_q, \ldots \).
Corollary 5.1. The pairing
\[ \Omega_U(Z) \otimes S \to \mathbb{Z}, \quad (\sigma, s) \mapsto s^H(\sigma) \]
induces an isomorphism of the ring \(\Omega_U(Z)\) with the Hopf algebra \(S^*\).

§6. The Landweber–Novikov algebra is the enveloping algebra of the Lie algebra of formal vector fields on the line

The result stated in the title of this section was obtained jointly by A. V. Shokurov and the author in 1978 ([3]). Here we give an exposition of this result convenient for the subsequent sections. Details about the Lie algebras of formal vector fields may be found in [5].

Consider the group \(G = \text{Diff}_1(Z)\) of formal diffeomorphisms of the line of the form
\[ x(t) = t + x_1 t^2 + \cdots + x_k t^k + \cdots, \quad x_k \in \mathbb{Z}, \]
in which multiplication is, of course, the composition of diffeomorphisms. By the coordinates of a diffeomorphism we mean the set of its coefficients, i.e., we put \(x = (x_1, \ldots, x_k, \ldots)\). Thus if \(x = (x_k)\) and \(y = (y_k)\) are two diffeomorphisms, then the coordinates of the diffeomorphism \(z = y \cdot x\) will be the coefficients of the series
\[ x(y(t)) = (t + \sum y_j t^j + \cdots) + \sum x_k (t + y_j t^j + \cdots)^k. \]

For the ring of functions on the group \(G\), let us take the polynomial ring \(P\) with integer coefficients in the coordinates \(x \in G\). To each cobordism class \(a \in \Omega_v^n\) we assign a function on the group \(G\). For any \(x \in G\) consider the multiplicative operation \(\varphi_x \in mS\) that takes the value \(\varphi_x(u)\) on a canonical element \(u \in U^2(CP(\infty))\) equal to the series \(x(u)\) and put \(\sigma(x) = e\varphi_x(\sigma)\). Thus we have defined the map \(\lambda: \Omega_U \to P\).

Theorem 6.1. (1) \(\lambda\) is a ring homomorphism and can be extended to a ring homomorphism \(\hat{\lambda}: \Omega_U(Z) \to P\).

(2) Suppose \(\{\alpha_k\}\) are the coefficients of the Chern–Dold character. Then we have \(\hat{\lambda}(\alpha_k)(x) = x_k\), i.e., the coefficients \(\alpha_k\) are taken to the coordinate polynomials on the group \(G\) and we have the isomorphism
\[ \Omega_U(Z) = \mathbb{Z}[\alpha_1, \ldots, \alpha_k, \ldots] \cong P \]
under which the action of the group of multiplicative operations on \(mS\) on the cobordism ring \(\Omega_U(Z)\) corresponds to the action of the group \(G\) by right translations on its function ring \(P\).

Proof. (1) Suppose \(\sigma_1, \sigma_2 \in \Omega_U\). Then
\[ \hat{\lambda}(\sigma_1 \sigma_2)(x) = e\varphi_x(\sigma_1 \sigma_2) = (e\varphi_x(\sigma_1))(e\varphi_x(\sigma_2)) = \hat{\lambda}(\sigma_1)(x) \cdot \hat{\lambda}(\sigma_2)(x). \]
Since \(e\varphi_x(\sigma) \in \mathbb{Z}\) for any \(\sigma \in \Omega_U(Z)\), it follows that \(\hat{\lambda}\) can be extended to a homomorphism \(\Omega_U(Z) \to P\).
We have \( \text{ch}_U u = t + \sum \alpha_k t^{k+1} = \alpha(t) \), therefore
\[
\varphi_*, \text{ch}_U u = t + \sum \varphi_*(\alpha_k) t^{k+1}.
\]
Using the fact that the Chern-Dold character \( \text{ch}_U u \) commutes with the action of the operations, we obtain
\[
(19) \quad \varphi_*, \text{ch}_U u = \text{ch}_U \varphi_*(u) = \text{ch}_U \left( u + \sum x_n u^{n+1} \right) = \alpha(t) + \sum x_n \alpha(t)^{n+1},
\]
i.e., the action of the operation \( \varphi_* \) on \( \alpha_k \) is given by the formula
\[
(20) \quad \varphi_*(\alpha_k) = \alpha_k + \sum x_n [\alpha(t)^{n+1}]_{k+1},
\]
where \([\alpha(t)]_k\) is the coefficient at \( t^{k+1} \) in the series \( a(t) \), and therefore
\[
\lambda(\alpha_k) = e \varphi_*(\alpha_k) = x_k.
\]
Thus we have proved that \( \lambda \) determines an isomorphism.

Now consider the representation of the group \( G \) on its ring of functions induced by right translations. According to (18), for any \( y \in G \) we have
\[
x(\lambda(\alpha_k))(y) = \lambda(\alpha_k)(yx) = \sum x_n [\alpha(t)^{n+1}]_{k+1} \]
\[
= \lambda \left( \alpha_k + \sum x_n [\alpha(t)^{n+1}]_{k+1} \right)(y).
\]
Using formula (20), we obtain
\[
x(\lambda(\alpha_k))(y) = \lambda(\varphi_*(\alpha_k)). \quad \Box
\]

The Lie algebra of the group \( G = \text{Diff}_1(\mathbb{Z}) \) is the Lie algebra of formal vector fields on the line \( L_1(1) \). The enveloping algebra \( \mathfrak{g} \) of the Lie algebra \( L_1(1) \) is the algebra of differential operators \( D_\omega, \omega \in \mathbb{Z}[t] \) whose action on the ring of functions \( a(x) \in P \) is given by the formula
\[
a(x \cdot y) = \sum_{\omega} D_\omega a(x) y^\omega.
\]
Therefore, \( \mathfrak{g} \) is the algebra of all left-invariant differential operators on the group \( G \).

Now Theorem 6.1 directly implies

**Corollary 6.1.** The map \( S \rightarrow \mathfrak{g}, s_\omega \mapsto D_\omega \) defines an isomorphism of the Landweber-Novikov algebra onto the enveloping algebra of the Lie algebra \( L_1(1) \) that induces an isomorphism of the S-module \( \Omega_U(\mathbb{Z}) \) with the g-module \( P \).

Suppose \( A^U(\mathbb{Z}) \) is the algebra of operations of the form \( \sum a_\omega s_\omega \), where the coefficients \( a_\omega \) are in \( \Omega_U(\mathbb{Z}) \).

**Corollary 6.2.** The algebra \( \Omega_U(\mathbb{Z}) \) is isomorphic to the algebra of all differential operators on the group \( \text{Diff}_1(\mathbb{Z}) \).

Novikov has proved that Theorem 6.1 implies
COROLLARY 6.3. Under the isomorphism $\Omega_U(\mathbb{Z}) \cong S^*$, the embedding $\Omega_U \subset \Omega_U(\mathbb{Z})$ induces the embedding $A_U \subset S^*_S$, i.e., $A_U$ is the operator double of the $S$-module $S^*$ with respect to the Milnor action of the Hopf algebra $S$ dual to the action of $S$ on itself by right translations.

Further we shall need a result from [3] that describes the image of the ring $\Omega_U$ in $P$ in terms of the group of diffeomorphisms.

DEFINITION 6.1. The Hirzebruch genus associated with the series
\[ x(t) = t + \sum x_k t^{k+1}, \quad x_k \in \mathbb{Q}, \]
is the ring homomorphism $L_x : \Omega_U \to \mathbb{Q}$ that to each cobordism class $[M^{2n}] \in \Omega_U^{-2n}$ assigns the value given by the formula
\[ L_x[M^{2n}] = \left( \prod_{q=1}^{n} \frac{t_q}{x(t_q)}, \langle M^{2n} \rangle \right), \]
where $M^{2n}$ is a smooth manifold whose stable tangent bundle $\tau(M^{2n})$ is a complex bundle with complete Chern class in cohomology
\[ C^H(\tau) = 1 + C_1^H(\tau) + \cdots + C_n^H(\tau) = \prod_{q=1}^{n}(1 + t_q) \]
and $\langle M^{2n} \rangle$ is the fundamental cycle in homology.

DEFINITION 6.2. The Todd genus $T$ is the Hirzebruch genus associated to the series $(1 - \exp(-t))$.

THEOREM 6.2. Let $\text{Diff}_1(\mathbb{Q})$ be the group of formal diffeomorphisms of the line over the field of rational numbers. Then the homomorphism $\text{Diff}_1(\mathbb{Q}) \to \mathbb{Q}$ taking each polynomial $\chi([M^{2n}]) \in P$ to its value on the diffeomorphism $x(t)$ coincides with the Hirzebruch genus $L_x[M^{2n}]$.

Proof. As shown above, the coefficients of the Chern–Dold character $\alpha_k$ are taken to the coordinate functions $\text{Diff}_1$ by the homomorphism $L_x$. Therefore it suffices to prove that $L_x(\alpha_k) = x_k$. But this immediately follows if we compare the formulas for $\text{ch}_U$ and $L_x$. \(\square\)

Let us identify the ring $P$ of integer polynomials on the group $\text{Diff}_1(\mathbb{Q})$ with the graded ring $\mathbb{Z}[\alpha_1, \ldots, \alpha_k, \ldots]$, \(\deg \alpha_k = -2n\).

Denote by $U^{-2n} \subset P$ the group of all homogeneous (of degree $-2n$) integer polynomials which remain an integer after a left shift by the diffeomorphism $T(t) = 1 - \exp(-t)$ and denote by $U$ the graded ring $\sum_{n \geq 0} U^{-2n}$.

COROLLARY 6.4. Under the identification of the ring $\Omega_U(\mathbb{Z})$ with the ring $P$, the ring of complex cobordisms $\Omega_U$ is identified with the ring $U$.

Proof. Suppose $a(x) \in U^{-2n}$. According to (21), we have
\[ T(a(x)) = a(T \cdot x) = \sum (D_x a)(T)x^a. \]
Now using Corollary 6.1 from Theorem 6.2, we obtain

\[ T(a(x)) = \sum T_{s_\omega}(a)x^\omega. \]

Therefore, if \( T(a(x)) \in P \), then \( T_{s_\omega}(a) \in \mathbb{Z} \) for all \( \omega \in \mathbb{Z}^\omega \). According to the Stong–Hattori theorem ([11]), the element \( a \in \Omega_{U-2n}(\mathbb{Z}) \) belongs to the ring \( \Omega_{U-2n} \) if and only if the Todd genus of the elements \( s_\omega(a) \) is an integer for all \( \omega \in \mathbb{Z}^\omega \). \( \square \)

§7. The semigroup of multiplicative operations in complex cobordisms is the semigroup of maps into itself of the group of formal diffeomorphisms of the line

For the group \( G = \text{Diff}_1(\mathbb{Z}) \) with the ring of functions \( P \), the map \( f : G \rightarrow G \) is called polynomial if \( a(f(x)) \) is a polynomial for any polynomial \( a(x) \in P \). It is clear that any such map is given by the following series

\[ f(x; t) = t + \sum f_k(x)t^{k+1}, \quad f_k(x) \in P. \]

Denote by \( G^G \) the set of all polynomial maps of the group \( G \) into itself. Then the semigroup \( G^G \) (see §1) corresponding to the action of \( G \) on itself by right translations is defined. Let us describe the multiplication in \( G^G \) explicitly. Let \( f_q \in G^G, q = 1, 2, \) and \( x \in G \). Then

\[ f_q(x; t) = t + \sum f_{q,k}(x)t^{k+1}, \]

\[ f_q(x,t) = t + \sum x_k t^{k+1}, \quad x_k \in \mathbb{Z}. \]

In \( G^G \), we have

\[ (f_2f_1)(x,t) = f_2(x,t)f_1(xf_2(x,t); t) \]

\[ = f_2(x,t) + \sum f_{1,k}(xf_2(x,t))f_2^{k+1}(x,t), \]

where \( xf_2(x,t) = x(t) + \sum f_{2,n}(x)x(t)^{n+1} \).

Now consider the ring \( \mathcal{U} \in P \) introduced in §6. Denote by \( G^G(\mathcal{U}) \) the subset of \( G^G \) consisting of the series

\[ f(x,t) = t + \sum f_k(x)t^{k+1}, \quad f_k(x) \in \mathcal{U}. \]

A direct verification shows that \( G^G(\mathcal{U}) \) is closed with respect to the multiplication in \( G^G \), and, therefore, the semigroup \( G^G(\mathcal{U}) \) is defined.

**Theorem 7.1.** The semigroup of multiplicative operations \( mA_U \) is isomorphic to the semigroup of maps \( G^G(\mathcal{U}) \).

**Proof.** Suppose \( \varphi_1, \varphi_2 \) are two multiplicative operations. Then they are determined by their values on the class \( u \in U^2(\mathbb{C}P(\infty)) \):

\[ \varphi_q u = \varphi_q(u) = u + \sum \varphi_{q,k}u^{k+1}, \quad \varphi_{q,k} \in \Omega_U, \quad q = 1, 2. \]
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Under the embedding \( \Omega_U \rightarrow \Omega_U(\mathbb{Z}) = \mathbb{Z}[\alpha_1, \ldots, \alpha_k, \ldots] \), we can identify \( \varphi_{q, k} \) with the polynomials \( \varphi_{q, k}(\alpha) \), where \( \alpha = (\alpha_1, \ldots, \alpha_k) \).

By the definition of multiplication in \( mA_U \), we have

\[
(\varphi_2 \varphi_1)u = \varphi_2 \varphi_1(u) = \varphi_2(u) + \sum \varphi_{q, k}(\varphi_2 \varphi_1)(\alpha)\varphi_2(u).
\]

Thus by assigning to each operation \( \varphi \in mA_U \) the series \( \varphi(u) = \varphi(\alpha; u) \), we obtain, according to Corollary 6.4, the bijection \( mA_U \cong G^G(U) \). Further,

\[
\varphi_2 \text{ch}_U(u) = t + \sum_{n \geq 0} (\varphi_2 \alpha_n)t^{n+1},
\]

\[
\text{ch}_U \varphi_2(u) = \alpha(t) + \sum \varphi_{q, k}\alpha(t)^{k+1},
\]

i.e., \( \varphi_2 \alpha = \alpha \varphi_2(\alpha; u) \). Therefore, comparing the multiplication formulas, we see that the semigroups \( mA_U \) and \( G^G(U) \) are isomorphic. \( \square \)

§8. The quantum double of the Landweber–Novikov algebra
is a subalgebra in the algebra of operations
of the doubled theory of complex cobordisms

The doubled theory of complex cobordisms (bordisms) is constructed for smooth manifolds whose stable tangent bundle \( \tau \) possesses a fixed splitting into two complex bundles \( \tau = \tau_0 \oplus \tau_1 \).

Suppose \( U_{n,m} = U(n) \times U(m) \) is the product of unitary groups. The standard embedding \( k^{n,m} : U_{n,m} \subset U_{n+k,m+q} \) induces a map of classifying spaces

\[
B_k^{n,m} : BU_{n,m} \rightarrow BU_{n+k,m+q}, \quad (B_k^{n,m})^*(\xi_{n+k} \times \xi_{m+q}) = \xi_n \times \xi_m \oplus 1_{k+q},
\]

and the corresponding map of Thom spaces

\[
M_k^{n,m} : M_{n+k,m+q} \cong M_n \wedge M_{m+q} \\rightarrow S^{2(k+q)}M_n \wedge M_m \cong S^{2(k+q)}M_{n,m}.
\]

Definition 8.1. The Thom spectrum in the theory of doubled complex cobordisms \( DU^*(\cdot) \) is the following sequence of Thom spaces together with their joining maps:

\[
DMU = \{ M_n, m = M_n \wedge M_m : M_k^{n,m} \}.\]

Thus we have

\[
DU^q(W, V) = \lim_{n, m} [S^{2(n+m)-q}(W/V), M_n, m],
\]

\[
DU^q(W, V) = \lim_{n, m} [S^{2(n+m)+q}, (W/V) \wedge M_n, m].
\]

Let us put, as usual,

\[
DU^q(W) = DU^q(W_+), \quad DU^q(W) = DU^q(W_-),
\]

\[
DU^*(W, V) = \sum DU^q(W, V), \quad DU^*(W, V) = \sum DU^q(W, V).
\]
In the case \( W = (\text{point}) \), by using the Thom isomorphism from the theory of complex cobordisms \( U, (\cdot) [11] \), we obtain
\[
\Omega_{DU}^{2q} = DU^*(\text{point}) = \lim_{n,m} \frac{S^{2(n+m+q)}}{M_n \wedge M_m} = \lim_{n,m} \frac{U_{2(n+m+q)}(M_n) \cong U_{2q}(BU)}{M_n \wedge M_m}.
\]

Multiplication in the theory \( DU^*(\cdot) \) is defined by means of the map of spectra \( D_M : (DMU) \times (DMU) \to DMU \) induced by the sequence of canonical maps \( M_{n,m} \wedge M_{k,q} \to M_{n+k,m+q} \). It is easy to verify that the isomorphism \( \Omega_{DU}^{2q} \cong U_{2q}(BU) \) can be extended to an isomorphism of rings \( \Omega_{DU} \cong U_*(BU) \), where in \( U_*(BU) \) one takes the multiplication in the bordisms of the \( H \)-space \( BU \) induced by the multiplication \( BU \times BU \to BU \) (via the Whitney sum).

If we assign to each manifold \( M^{2n} \) with split stable complex tangent bundle \( \tau = \tau_1 \oplus \tau_2 \), its complex cobordism class, we obtain a multiplicative transformation \( \pi : DU^*(\cdot) \to U^*(\cdot) \). The assignments
\[
\tau_1 : (M^{2n}, \tau) \to (M^{2n}, \tau \oplus 0); \quad \tau_2 : (M^{2n}, \tau) \to (M^{2n}, 0 \oplus \tau)
\]
induce the corresponding multiplicative transformations
\[
\tau_1^*, \tau_2^* : U^*(\cdot) \to DU^*(\cdot).
\]
A direct verification yields the following

**Lemma 8.1.** The multiplication \( \mu \) in the theory of complex cobordisms splits into the composition of homomorphisms
\[
\tau_1 \otimes \tau_2 : U^*(\cdot) \otimes U^*(\cdot) \to DU^*(\cdot) \otimes DU^*(\cdot) \to U^*(\cdot).
\]

Now let us describe the algebra of stable cohomology operations \( A_{DU} \) in the theory \( DU^*(\cdot) \). In the role of the universal Thom class \( u_n, m \in DU^{2(n+m)}(M_n, m) \), let us choose the class \( (u_n, m) \) and put \( s_{0,n}^\omega = u_n, m \in C_\omega(\xi_n) \), \( s_{0,n}^\omega = u_n, m \in C_\omega(\xi_n) \).

It can be verified in a standard way that the sequences \( \{s_{0,n}^\omega\} \) and \( \{s_{0,n}^\omega\} \) determine the stable cohomology operations \( s_0^\omega, s_0^\omega \in A_{DU}, \omega \in \mathbb{Z}^\infty \). The operations \( s_{0,0} = s_{0,0}^\omega \) and \( s_{0,0} = s_{0,0}^\omega \) commute for all \( \omega_1 \) and \( \omega_2 \).

Denote by \( DS \) the free topological commutative group with topological basis \( s_0^\omega, s_0^\omega, \omega_1, \omega_2 \in \mathbb{Z}^\infty \). We obtain
\[
\begin{align*}
(1) & \text{ DS is a subalgebra of } A_{DU}; \\
(2) & \text{ the algebra } DS \text{ is isomorphic to the tensor product of Hopf algebras } S^l \otimes S^r, \\
& \text{ where } S^l \cong \tau_1 S \text{ and } S^r \cong \tau_1 S. \\
(3) & A_{DU} \cong \Omega_{DU}(S^l \otimes S^r). \\
\end{align*}
\]

Thus it only remains to describe the representation of the algebra \( S^l \otimes S^r \) in the ring \( DU^*(\cdot) \).

Denote by \( m A_{DU} \) the semigroup of all multiplicative operations in the theory \( DU^*(\cdot) \). Using, as usual, the identification of the ring \( U^*(MU(n)) \) with its image in \( U^*(\mathbb{CP}(\infty)) \) under the monomorphism \( h^* S_0 \), we obtain
LEMMA 8.2. Let \( u, u_r \) be the Thom classes \( u_{1,0}, u_{0,1} \) from \( DU^2(\mathbb{C}P(\infty)) \). Each operation \( \psi \in mA_{DU} \) is determined by its values on the classes \( u_{1,0} \) and \( u_{0,1} \), i.e., by the series

\[
\psi u_l = \psi_l(u_l) = u_l + \sum \psi_{l,k} u_l^{k+1}, \quad \psi u_r = \psi_r(u_r) = u_r + \sum \psi_{r,k} u_r^{k+1},
\]

where \( \psi_{l,k}, \psi_{r,k} \in \Omega_{DU} \).

Further we shall need the following fundamental result from the theory of complex cobordisms (see [7, Appendix 1]).

Consider the tensor product of one-dimensional complex universal bundles \( \xi_l' \otimes \xi_r' \to \mathbb{C}P(\infty) \times \mathbb{C}P(\infty) \). Then the following results hold.

1. The Chern class

\[
C_1(\xi_l' \otimes \xi_r' \in \mathbb{C}P(\infty) \times \mathbb{C}P(\infty)) \cong \Omega_U[[u, v]],
\]

where \( u = C_1(\xi_l'), v = C_1(\xi_r') \) is determined by the formal series

\[
C_1(\xi_l' \otimes \xi_r') = F(u, v) = u + v + \sum a_{k, q} u^k v^q,
\]

which defines the formal group over \( \Omega_U \).

2. The coefficients of the series \( a_{k, q}, \deg a_{k, q} = -2(k + q - 1) \) generate the ring \( \Omega_U = \mathbb{Z}[a_1, \ldots, a_n, \ldots] \), \( \deg a_n = -2n \).

Now let us present a similar description of the ring \( \Omega_{DU} \). The universal bundle \( \xi_1 \to \mathbb{C}P(\infty) \) has two canonical first Chern classes in the theory \( DU^*(\cdot) \):

\[
u_l = C_{1,l}(\xi_l') = \nu_l C_1(\xi_1), \quad \nu_r = C_{1,r}(\xi_r') = \nu_r C_1(\xi_1).
\]

LEMMA 8.3. Over the ring \( \Omega_{DU} \), two formal groups are defined

\[
C_{1,l}(\xi_l' \otimes \xi_r') = F_l(u_l, v_l) = u_l + v_l + \sum a_{k, q} u_l^k v_l^q, \quad a_{k, q} \in \mathrm{Im} \nu_l \Omega_U,
\]

\[
C_{1,r}(\xi_l' \otimes \xi_r') = F_r(u_r, v_r) = u_r + v_r + \sum a_{k, q} u_r^k v_r^q, \quad a_{k, q} \in \mathrm{Im} \nu_r \Omega_U,
\]

as well as the series

\[
u_l = \varphi(u_l) = u_l + \sum \varphi_k u_l^{k+1},
\]

that determines a strong isomorphism of these groups, where \( \pi \varphi_k = 0 \).

PROOF. Since \( DU^*(\cdot) \) is a multiplicative cohomology theory, for any class \( u \in DU^2(\mathbb{C}P(\infty)) \) such that \( \varepsilon u \) is a generator of the group \( H^2(\mathbb{C}P, \mathbb{Z}) = \mathbb{Z} \), we have the isomorphism \( DU^*(\mathbb{C}P(\infty)) \cong \Omega_{DU}[[u]] \). Therefore,

\[
DU^*(\mathbb{C}P(\infty)) \cong \Omega_{DU}[[u]] \cong \Omega_{DU}[[u]]
\]

and so \( u_l = \varphi(u_l) = u_l + \sum \varphi_k u_l^{k+1} \).
Now consider the map $\gamma : CP(\infty) \times CP(\infty) \to CP(\infty)$ given by $\gamma^*\xi_1 = \eta_1^* \otimes \eta_1''$. We have

$$F_1(\varphi(u), \varphi(v)) = C_{1,1}(\eta_1^* \otimes \eta_1'') = \gamma^*C_{1,1}(\xi_1) = \gamma^*\varphi(C_{1,1}(\xi_1)) = \varphi(C_{1,1}(\gamma^*\xi_1)) = \varphi(C_{1,1}(\eta_1^* \otimes \eta_1'')) = \varphi(F_1(u, v)).$$

Now denote by $\Omega_{U,1} = \mathbb{Z}[a_+^1], \Omega_{U,r} = \mathbb{Z}[a_+^r]$ the subrings of $\Omega_{DU}$ generated by the coefficients of the formal groups $F_1(u, v)$ and $F_r(u, v)$ and by $B = \mathbb{Z}[\varphi]$ the subring of $\Omega_{DU}$ generated by the coefficients of the series $\varphi(u)$. It follows from the previous computation that in the ring $\Omega_{DU}$, up to decomposable elements, we have the relation

$$a_{k+q} - a_{k,q} = \binom{k+q}{k}\varphi_{k+q-1},$$

where $\binom{k+q}{k}$ is the binomial coefficient. Hence the rings $\Omega_{U,1}$ and $\Omega_{U,r}$ together do not generate the ring $\Omega_{DU}$.

The proof of the next result is left to the reader.

**Theorem 8.1.** We have the following isomorphisms

$$\Omega_{DU} \cong \mathbb{Z}[a_1^1, \varphi_1] \cong \mathbb{Z}[a_1^r, \varphi_r].$$

**Lemma 8.4.** The subring $B = \mathbb{Z}[\varphi] \subset \Omega_{DU}$ is closed with respect to the action of the algebra $S^s \otimes S^r$ and is therefore a Milnor module over the Hopf algebra $S \otimes S$.

**Proof.** Consider the universal multiplicative cohomology operations

$$S^s, S^r : DU^*(V) \to DU^*(V)[[t_k, \tau_q]]$$

such that

$$S^s u_1 = u_1 + \sum t_k u_1^{k+1} = S^s(u_1), \quad S^r u_r = u_r, \quad S^s u_r = u_r + \sum \tau_q u_r^{q+1} = S^r(u_r).$$

Here $t_k$ and $\tau_q$ are formal variables. It follows from the universality of the Thom class $u_{n,m}$ that

$$S^s u_{n,m} = \sum s_{n,m}^t t^n, \quad S^r u_{n,m} = \sum s_{n,m}^r \tau^m,$$

i.e., $S^s$ and $S^r$, as elements of the algebra $S^s \otimes S^r[[t_k, \tau_q]]$, are the generating series for all the operations in $S^s \otimes S^r$.

For $V = CP(\infty)$ we have

$$S^s(u)_1 = S^s u_1 = S^s u_r + \sum (S^s\varphi_s)(S^s u_r)^{s+1} = u_r + \sum (S^s \varphi_s) u_r^{n+1}.$$ 

Therefore, setting $u_r = t$, we obtain

$$t + \sum (S^s \varphi_s) t^{n+1} = S^s(\varphi(t)) = (\varphi S^s)(t),$$

(26)
i.e., the operations $s^j$ act on $\varphi_n$ by right translations. Further,

$$S'u = S'u + \sum (S'\varphi_n)(S'u)^{n+1}. $$

Therefore, setting $u_t = t$ again, we obtain

$$\varphi(t) = S'(t) + \sum (S'\varphi_n)(S'(t))^{n+1}, $$

i.e.,

$$t + \sum S'\varphi_n t^{n+1} = \varphi((S')^{-1}(t)) = ((S')^{-1}\varphi)(t), $$

so that the operations $s^j_{n'}$ act on $\varphi_n$ by left translations.

**Corollary 8.1.** The operator double $B(S \otimes S)$ associated with the Milnor action of $S \otimes S$ on $B$ is a subalgebra of $A_{DU}$.

Now everything is ready to obtain the result appearing in the heading of the present section.

**Theorem 8.2.** Suppose $S^* = \mathbb{Z}[s^{(1)}, \ldots, s^{(q)}, \ldots]$ is the Hopf algebra dual to the Hopf algebra $S$. Then the isomorphism $S^* \rightarrow B$, $s^{(q)} \mapsto \varphi_q$ together with the diagonal homomorphism $\Delta: S \rightarrow S \otimes S$ determine an embedding of the quantum double $D(S)$ in $B(S \otimes S) \subset A_{DU}$.

**Proof.** Formulas (26), (27) show that the diagonal homomorphism $\Delta: S \rightarrow S \otimes S$ induces the action $\text{ad}$ of the Hopf algebra $S$ on $B^* \otimes S$. Therefore $BS \cong S^*_{ad}$. According to Theorem 3.2, we have the isomorphism $D(S) \cong S^*_{ad}$. Thus the embedding $BS \subset B(S \otimes S)$ defines an embedding of the quantum double $D(S)$ of the Landweber–Novikov algebra into $A_{DU}$. □

**References**


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