Topics in Topology and Mathematical Physics

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The S. P. Novikov Seminar

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Our seminar came into being in the mid-sixties. S. P. Novikov's works on the topology of foliations and on the topological invariance of rational Pontryagin classes were concluded in 1965. A new stage of research was about to begin, devoted to extraordinary (generalized) cohomology theories: complex cobordisms and $K$-theory. Novikov's older pupils (e.g. V. L. Golo) had moved on to other fields.

By that time $K$-theory had become extremely popular in wide circles of the mathematical community in connection with the work of Atiyah and Singer on the index of elliptic operators; besides, Adams discovered brilliant applications of $K$-theory in topology itself: the number of linearly independent vector fields on spheres was found, important subgroups in the stable homotopy groups of spheres were computed (the image of Whitehead's $J$-homomorphism). The possibility of constructing regular methods, based on $K$-theory, for the computation of homotopy groups (in particular those of spheres), more effective than the classical methods of Cartan-Serre-Adams, was being discussed (the Adams program).

Among the participants of the seminar during the first two-three years, the names of V. Buchstaber, A. Mishchenko, I. Bernstein, I. Volodin, S. Smirnov, S. Vishik, and F. Bogomolov should be singled out.

In 1966–67 S. P. Novikov [49] carried out his own program, creating methods for the regular computation of stable homotopy groups on the basis of complex cobordisms. It turned out that the Adams program for creating such methods on the basis of $K$-theory could not be realized in principle. In contrast, the new techniques of complex cobodisms were extremely rich in ideas and applications to topology and algebra, namely:

- formal groups, together with their applications in homotopy theory as well as in the study of fixed points of finite and compact transformation groups of smooth manifolds, including the remarkable properties of elliptic genera ([50, 11, 5, 31, 34, 59, 38]) discovered much later;
- the theory of multivalued formal groups, together with its applications to topology, algebra, analysis, including the relationship with generalized shift operator theory ([12, 6, 7, 8]);
- the algebra of operations from complex cobordism theory with its numerous topological applications and beautiful intrinsic algebraic structure that later led to the "operator (Heisenberg) double" of Hopf algebras, which is the quantum analog of the algebra of differential operators on a Lie group ([49, 13, 58, 9]).
These ideas are developed in the article by V. Buchstaber which also contains new results in this direction.

By the end of the sixties, the participants of the seminar, as well as its subject matter, changed noticeably. The research interests of several participants, e.g. I. Bernstein and S. Vishik, shifted to other fields of mathematics. A whole new generation of extremely talented young researchers appeared in the seminar: G. Kasparov, O. Bogoyavlenskii, S. Gusein-Zade, S. Brakhman, I. Krichever, V. Dubrovin, M. Bruk, V. Krasnov, M. Brodskii, S. Tankeev, F. Zak, R. Nadiradze, A. Peretsetskii, A. Shokurov, N. Panov, V. Vedenyapin, and others. For about two years, B. G. Moishezon became co-director of the seminar. During this period its participants studied Kahler and algebraic geometry. Later several members of the seminar began working on the problems of algebraic geometry itself.

Among the participants were several others, who, beginning in 1974, greatly contributed to the application of methods coming from algebraic geometry to modern mathematical physics (more specifically, to the periodicity problems in soliton theory and in integrable systems). The work of these participants, carried out in the seventies and eighties, became widely known ([24, 52, 35, 41, 17, 23]. In this volume these ideas are developed in the papers by I. Krichever, S. Natanzon, A. Veselov. A striking example of feedback, namely, the application of the theory of nonlinear equations to algebraic geometry, was Novikov's conjecture on the characterization of the Jacobians of Riemann surfaces (the Riemann-Schottki problem). The first notable advance in this problem was the work of B. Dubrovin [18]. A complete proof was given by Shiota [61].

During the second half of the sixties, several participants of the seminar also studied the topology of non-simply-connected manifolds, developing the ideas that had arisen in foliation theory, as well as those appearing in the proof of the topological invariance of the rational Pontryagin classes. At the time, the final formulation of S. P. Novikov's conjecture on higher signatures [51] for manifolds with arbitrary fundamental group was put forward and was established for abelian fundamental groups $\pi_1$. In this period the seminar, besides its traditional contacts with the seminars of the leading Moscow mathematical schools, was in constant very intense interaction with the V. A. Rokhlin seminar in Leningrad.

This period also witnessed the appearance of ideas on the Hermitian analogs of algebraic $K$-theory, based on the language and basic concepts of the Hamiltonian formalism, an algebraic version of sorts of symplectic geometry [51].

The $K$-theory of infinite-dimensional complexes was then constructed, leading up to complete answers in many important cases, including computations for the classifying spaces of compact Lie groups and Eilenberg-MacLane complexes [10]. The correct higher analogs of algebraic $K$-theory $K^0$, $K^1$ (at the same time as Quillen, but on the basis of a different idea) were obtained [66].

Soon after that, the development of Fredholm representations was undertaken, both for the construction of a topological $K$-homology theory and for the higher signature problem ([32, 47]).

The S. P. Novikov conjecture on higher signatures eventually became widely known in mathematics. A huge number of papers is devoted to this conjecture. It gave the impetus for finding deep relationships between topology, algebra, and functional analysis ([48, 33, 16, 14, 15]). A conference was held on this topic in
Oberwolfach in September 1993. Two volumes of the proceedings of the conference [70] will include both research and survey papers illustrating the current status of the conjecture.

At the beginning of the seventies, the research interests of the participants of the seminar diverged: new seminars were organized, where the branches of topology and algebra mentioned above (cobordisms, formal groups, problems of nonsimply-connected manifolds, including Hermitian $K$-theory, problems of higher signatures and functional analysis techniques) were still studied, e.g. at the V. Bachstaber seminar or the A. Mishchenko seminar.

In the second half of the seventies, only one pure topologist began research at the S. P. Novikov seminar itself on the then very new subjectmatter related to Sullivan's ring approach to rational homotopy type: this was I. Babenko ([2, 3]).

Around 1970–71, the seminar concentrated on the study of special and generalized relativity (at the time one of the co-directors of the seminar was V. P. Myasnikov).

At the beginning of 1971, S. P. Novikov began working at the Landau Institute of Theoretical Physics and the interests of the seminar shifted more and more towards the mathematical problems of modern theoretical physics. Jointly with several of his pupils (O. Bogoyavlenskii, B. Dubrovin, I. Krichever), S. P. Novikov originated the qualitative theory of homogeneous cosmological models, solved the periodic problems of soliton theory, developed the theory of one-dimensional and two-dimensional Schrödinger operators in periodic electric and magnetic fields ([4, 24, 22, 53]). Later these ideas were developed and led to the creation of the theory of two-dimensional periodic and rapidly decreasing operators ([57, 29, 30, 37, 39]).

In the late seventies and in the eighties, new participants of the seminar joined in these directions of research, in particular A. Veselov, I. Taimanov, P. Grinevich, O. Mokhov, S. Tsarev, A. Lyskova, R. Novikov [26].

In the process of solving problems of physical nature, S. P. Novikov returned to topology: he found curious topological characteristics of the typical two-dimensional Schrödinger operator in a magnetic field and in a periodic lattice, that was later to play a crucial role in explaining the quantum Hall effect, initiated the notion of multivalued variational calculus in theoretical physics and mathematics, and constructed the analog of Morse theory for multivalued functions and functionals ([53, 54, 55, 56, 62]). A. Lyskova and I. Taimanov took part in these studies. The topic was later developed by quite a few researchers.

These topics were those where a whole new generation of the seminar's participants, working in topology, began their research: F. Voronov, A. Zorich, A. Lazarev, D. Millionshchikov, A. Alaniya, Le Tu Thang, I. Dynnikov, and others ([68, 67, 69, 43, 45, 46, 1, 44, 28]). In this volume these ideas are developed in the joint article by P. Grinevich and S. P. Novikov and the one by I. Dynnikov.

A curious cycle of new ideas in Euclidean geometry arose in the eighties as the result of the interaction with young theoretical physicists from the Landau Institute (Levitov, Kitaev, Kalugin): the beautiful concept of quasi-crystalline subgroup of the isometry groups of Euclidean space in the sense of Novikov–Veselov and other aspects of the geometry of quasi-crystals were successfully exploited by S. Piunikhin, V. Sadov, and Le Tu Thang [60]; the asymptotic problems of soliton theory led to the construction of a Hamiltonian theory of systems of hydrodynamic type (i.e., quasi-
linear systems of the first order), which had not appeared in the entire hundred-year history of this field. The Hamiltonian formalism of “hydrodynamic type” was discovered by S. P. Novikov and B. Dubrovin in 1983, giving rise to a new branch of Riemannian geometry ([27, 25, 26, 19]).

In the framework of this geometry, S. Tsarev found a method for integrating such systems ([63, 64, 65]). As the result of a cycle of further studies of several participants of the seminar (V. Avilov, S. P. Novikov, I. Krichever, G. Potemin), the complete analytic solution for the dispersive analog of the wave equation, undertaken in the early seventies by leading members of the Landau school (A. B. Gurevich and L. P. Pitaevskii), was finally obtained. The algebraic geometry realization of the Flaschke–MacLaughlin mean of the (Whitham) soliton equations was developed by Krichever [36, 40].

These geometric ideas play an important role in the construction of the now very popular two-dimensional topological quantum field theory [20], [21]. In the present volume this cycle of ideas appears in the articles by O. Mokhov, E. Feropontov, and in the subsequent one in those of B. Dubrovin, M. Pavlov, V. Alekseev.

In the eighties, several participants of the seminar made significant contributions to the development of the geometry and topology of supermanifolds. In particular, F. Voronov and A. Zorich constructed the correct superanalog of the de Rham complex [67]. The ideas of supersymmetry are developed in this volume in the article by F. Voronov.

In connection with the operator quantization of boson strings and two-dimensional conformal field theory, S. P. Novikov and I. Krichever defined the correct analogs of the Laurent and Fourier series for the expansion of meromorphic tensor fields of any weight on Riemann surfaces. The operator quantization program for strings was actively developed by physicists in 1970–74 (Veneciano, Virasoro, Mandelstam, and others) for surfaces of genus 0. The absence of analogs of Laurent–Fourier series on surfaces of positive genus stopped further development of this topic in the mid-seventies.

Using a different approach (the continual integral), A. Polyakov succeeded in quantizing the string for Riemann surfaces (diagrams) of any genus.

The implementation of the program of operator or algebraic quantization of strings became possible as the result of the work of S. P. Novikov and I. Krichever at the end of the eighties. The “Krichever–Novikov bases” and the “Krichever–Novikov algebra” that they constructed are the Riemann analogs of the Laurent–Fourier bases and of the Virasoro algebra [42]. This topic is that of numerous papers by many authors. In this volume it is represented by O. Sheinman’s article.

References


Translated by A. B. SOSSINSKY