On GLP-spaces

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Abstract

The following is a draft collection of miscellaneous preliminary results on GLP-spaces, the natural topological models for Japaridze's polymodal provability logic. The collection is poorly structured, lacks a proper introduction and references, and is generally not intended for publication in the current form.

1 GLP-algebras and spaces

We study topological models of a system of polymodal logic **GLP** due to Giorgi Japaridze.

Consider the language of propositional polymodal logic with modalities $[n]$ and $\langle n\rangle$ labeled by natural numbers. The system **GLP** is given by the following axiom schemata and rules.

Axioms: (i) Boolean tautologies;

(ii) $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$;

(iii) $[n]([n]\varphi \to \varphi) \to [n]\varphi$;

(iv) $[m]\varphi \to [n]\varphi$, for $m < n$;

(v) $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for $m < n$.

Rules: modus ponens, $\varphi \vdash [n]\varphi$.

We consider poly-topological spaces of the form $(X; \tau_0, \tau_1, \dots)$ where τ_i are topologies on a set X. The topological interpretation of modality $\langle n\rangle$ here is the derived set operator $\mathbf{D}_n$ corresponding to τ_n: for a subset $A \subseteq X$ we define $x \in \mathbf{D}_n(A)$ iff for all τ_n-open U containing x, there is a $y \neq x$ such that $y \in U \cap A$.

Every such space gives rise to its dual boolean algebra X^* of subsets of X equipped with unary operators

$$\langle n\rangle : A \longmapsto \mathbf{D}_n(A),$$

for each $n < \omega$.

Definition 1.1 A boolean algebra with operators $(\mathcal{A}; \langle 0\rangle, \langle 1\rangle, \dots)$ is a **GLP**-*algebra*, if it satisfies all the identities of the system **GLP**, in other words, if for each propositional formula $\varphi(\vec{x})$

$$\mathcal{A} \models \forall \vec{x}\, (\varphi(\vec{x}) = \top) \iff \mathbf{GLP} \vdash \varphi.$$

Definition 1.2 A poly-topological space X is a **GLP**-*space* if the following three conditions are satisfied:

(i) The space $(X; \tau_0)$ is scattered, that is, every $A \subseteq X$ has a τ_0-isolated point;

(ii) For all $m < n$, $\tau_m \subseteq \tau_n$;

(iii) For all $m < n$ and all $A \subseteq X$, $\mathbf{D}_m(A)$ is open in τ_n.

Notice that if τ_0 is scattered, so is each of the stronger topologies τ_n on X, for all $n < \omega$.

Theorem 1 *X is a* **GLP**-*space iff X^* is a* **GLP**-*algebra.*

Proof. As shown by Leo Esakia, validity of Löb's axiom (iii) is equivalent to scatteredness, and Conditions (ii) and (iii) correspond to Axioms (iv) and (v) of **GLP**, respectively. ⊠

Main examples of **GLP**-algebras come from proof theory, where they have been introduced under the name *graded provability algebras* (see [3, 1]). Under one possible proof-theoretic interpretation, modalities $\langle n\rangle$ correspond to reflection principles of restricted logical complexity in arithmetic acting as operators on the Lindenbaum algebra of a formal theory T. Provability algebras provide our main motivation for studying **GLP**-spaces, however they are not considered in this paper. We shall define and study some natural examples of **GLP**-spaces below.

2 Basic facts on GLP-spaces

2.1 Generated GLP-space

The properties (ii) and (iii) express, for each n, that topology τ_{n+1} is sufficiently strong w.r.t. τ_n. This motivates the following definition.

Definition 2.1 Let $(X; \tau_0)$ be given. Define inductively a sequence of topologies τ_n on X by setting that, for each $n < \omega$, τ_{n+1} is generated by the subbase $\tau_n \cup \{\mathbf{D}_n(A) : A \subseteq X\}$. We say that the poly-topological space $(X; \tau_0, \tau_1, \dots)$ is *generated from* $(X; \tau_0)$.

Proposition 2.2 *Let $(X; \tau_0)$ be a scattered space and let $\mathcal{X} := (X; \tau_0, \tau_1, \dots)$ be generated from $(X; \tau_0)$. Then $\mathcal{X}$ is a* **GLP***-space.*

Proof. Easy. ⊠

2.2 Separation properties

Scattered spaces are well-known to satisfy a weak form of separation located between T_0 and T_1.

Definition 2.3 A space $(X; \tau)$ is T_d if it satisfies one of the three equivalent conditions:

(i) Every point is an intersection of a closed and an open set;

(ii) For each $A \subseteq X$, $\mathbf{D}(A)$ is closed;

(iii) For each $A \subseteq X$, $\mathbf{D}(\mathbf{D}(A)) \subseteq \mathbf{D}(A)$.

Proposition 2.4 (Esakia) *Every scattered space is T_d.*

Proof. On a modal-logical level this theorem corresponds to a derivation of the transitivity principle $\Diamond\Diamond\varphi \to \Diamond\varphi$ from Löb's principle, a theorem due to Dick de Jongh. ⊠

Hence, every topology of a **GLP**-space is T_d. In general, scattered spaces need not be hausdorff or even T_1.

Example 2.5 Let $(X, \prec)$ be a partial ordering. The *upset topology* on $(X, \prec)$ is defined by the collection of upwards closed sets. It is an Alexandroff topological space and is scattered iff the ordering $(X, \prec)$ is upwards well-founded. However, for any nontrivial $(X, \prec)$, this topology is not T_1.

For topologies τ_1, τ_2, ... in a **GLP**-space one can, however, infer a bit more separation.

Proposition 2.6 *Let $\mathcal{X}$ be a* **GLP***-space. Then each τ_n, for $n \geqslant 1$, has to be T_1.*

Proof. It is sufficient to show that τ_1 is T_1. Let $a, b \in X$, $a \neq b$. We must show that there is an open set A such that $a \in A$ and $b \notin A$. Consider the set $A := \mathbf{D}_0(\{b\})$, which must be open in τ_1 and hence clopen in τ_1, since it is closed in τ_0. If $a \in A$ then a and b are separated by this clopen set, because $b \notin \mathbf{D}_0(\{b\})$. Otherwise, if $a \notin A$, a does not belong to the closure of $\{b\}$ (which is simply $\{b\} \cup \mathbf{D}_0(\{b\})$). It follows that the complement of $\{b\} \cup A$ is the required open set. ⊠

The following example shows that, in general, τ_1 need not be hausdorff.

Example 2.7 Let $(X, \prec)$ be a strict partial ordering on the set $X := \omega \cup \{a, b\}$ such that $m \prec n \iff m > n$ and $a \prec n$, $b \prec n$, for all $m, n \in \omega$. Let τ_0 be the upset topology on $(X, \prec)$ and let $\mathcal{X}$ be the **GLP**-space generated by (X, τ_0). Notice that for the upset topology

$$\mathbf{D}_0(A) = \{x \in X : \exists y \in A \ x \prec y\}.$$

Hence, sets of the form $\mathbf{D}_0(A)$ are downward closed. Thus, if A intersects ω, then $\mathbf{D}_0(A)$ contains an end-segment of ω. Otherwise, $\mathbf{D}_0(A) = \varnothing$. It follows that a base of open neighborhoods of a in τ_1 consists of sets of the form $I \cup \{a\}$ where I is an end-segment of ω. Similarly, sets of the form $I \cup \{b\}$ are a base of open neighborhoods of b. This space is clearly not hausdorff.

Below we shall study in some detail the GLP-space generated by the upset topology on an ordering $(\lambda, >)$, where λ is an ordinal. We note that, for this space, τ_1 will be the usual interval topology on λ, hence hausdorff.

2.3 Nontriviality conditions

Definition 2.8 A GLP-space $(X; \tau_0, \tau_1, \dots)$ is called *trivial*, if τ_1, and hence all τ_n for $n \geqslant 1$, are discrete.

Here we show that, in a nontrivial GLP-space, either τ_0 is non-hausdorff, or (X, τ_0) has to be rather large.

Recall that a topological space X is first-countable if every point $x \in X$ has a countable basis of open neighborhoods.

Lemma 2.9 *Suppose X is hausdorff and first-countable. For all $x \in \mathbf{D}(X)$ there is a subset A such that $\mathbf{D}(A) = \{x\}$.*

Proof. Select a countable basis of open neighborhoods of x, $(U_n)_{n\geqslant 0}$. We can additionally assume that $U_n \subset U_m$ whenever $n > m$. Select an element $u_n \in U_n \setminus \{x\}$, for each $n \geqslant 0$, using the fact that $x \in \mathbf{D}(X)$. Let $A := \{u_n : n \geqslant 0\}$.

Clearly, $x \in \mathbf{D}(A)$. If V is a neighborhood of x, there is an n such that $x \in U_n \subseteq V$. Then $u_n \in V \setminus \{x\}$ and $u_n \in A$.

We show that if $y \neq x$ then $y \notin \mathbf{D}(A)$. If $y \neq x$ select the neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$. Let $U_m \subseteq U$. Then, for all $n \geqslant m$ $u_n \in U$, hence $u_n \notin V$. Hence, $V \cap A$ is finite. Since X is hausdorff, one can select a smaller neighborhood $V' \subset V$ such that $V' \cap A = \varnothing$. $\boxtimes$

As an immediate corollary we obtain

Proposition 2.10 *For any GLP-space $\mathcal{X}$, if τ_0 is hausdorff and first countable, then $\mathcal{X}$ is trivial.*

This follows from the fact that all sets of the form $\mathbf{D}(A)$ in a GLP-space must be 1-open.

As a contrast to this proposition we remark that one obtains many examples of of nontrivial *countable* non-hausdorff GLP-spaces generated by upset topology on various well-founded partial orderings (e.g., on the one given in Example 2.7, but also on any countable well-ordering).

In the next section we define non-trivial hausdorff GLP-spaces generated by the standard interval topology of a well-ordering.

2.4 Ordinal spaces

Ordinal spaces provide perhaps the most natural examples of **GLP**-spaces.

Definition 2.11 Let κ be an ordinal. Let $\mathcal{O}(\kappa)$ denote the set of all ordinals $\alpha < \kappa$ equipped with the *interval topology* τ_0 generated by the standard base consisting of intervals of the form (α, β) where $\alpha < \beta$, $\alpha, \beta \in \mathcal{O}(\kappa) \cup \{\pm\infty\}$. $\mathcal{O}(\kappa)$ will also denote the GLP-space generated from τ_0. We call such GLP-spaces *ordinal spaces.*

For ordinal spaces we would like to characterize the topologies τ_1, τ_2, etc. in the more familiar terms. We would also like to know for which ordinals (if any) such spaces are non-trivial. Then we shall consider the question of topological completeness of the system **GLP** w.r.t. ordinal spaces.

We infer from Proposition 2.10 the following corollary.

Corollary 2.12 *For any countable κ, $\mathcal{O}(\kappa)$ is trivial.*

We will, in fact, show that for any uncountable κ $\mathcal{O}(\kappa)$ is non-trivial. Before going any further we would like to characterize some simpler bases for generated GLP-spaces.

3 Bases for GLP-spaces

Suppose $(X; \tau_0, \tau_1, \dots)$ is a GLP-space generated from $(X; \tau_0)$.

Definition 3.1 Let $\mathcal{B}_0$ be a base for τ_0. $\mathcal{B}_{n+1}$ is obtained from $\mathcal{B}_0$ by adding all sets of the form

$$A_0 \cap \mathbf{D}_k(A_1) \cap \mathbf{D}_k(A_2) \cap \cdots \cap \mathbf{D}_k(A_m) \tag{$*$}$$

such that $A_0 \in \mathcal{B}_0$, $k \leqslant n$, $A_i \subseteq X$, for $i \leqslant m$.

Proposition 3.2 *$\mathcal{B}_n$ is a base for τ_n, for each n.*

Proof. We reason by induction on n using the following lemma.

Lemma 3.3 *In any GLP-space, if $k < n$ then*

$$\mathbf{D}_n(A \cap \mathbf{D}_k(B)) = \mathbf{D}_n(A) \cap \mathbf{D}_k(B).$$

Proof. This corresponds to a well-known identity

$$(\langle n \rangle p \wedge \langle k \rangle q) \leftrightarrow \langle n \rangle (p \wedge \langle k \rangle q),$$

provable in modal logic **GLP**. $\boxtimes$

We shall obtain a base for τ_{n+1} by considering finite intersections of the form

$$Y \cap \mathbf{D}_n(B_1) \cap \mathbf{D}_n(B_2) \cap \cdots \cap \mathbf{D}_n(B_s), \tag{$**$}$$

where $Y \in \mathcal{B}_n$. By induction hypothesis, we can assume Y has the form $(*)$, for some $k < n$. If $s \geqslant 1$, the previous lemma allows one to get rid of all terms of the form $\mathbf{D}_k(A_i)$ replacing $\mathbf{D}_k(A_i)$ by $\mathbf{D}_n(B_1 \cap \mathbf{D}_k(A_i))$, the latter being equal to $\mathbf{D}_n(B_1) \cap \mathbf{D}_k(A_i)$. Hence, $(**)$ belongs to $\mathcal{B}_{n+1}$. $\boxtimes$

Remark 3.4 The base $\mathcal{B}_n$ can be further narrowed down a bit by considering only those sets $(*)$ where each $A_i \subseteq A$. This follows from the identity

$$I \cap \mathbf{D}(A) = I \cap \mathbf{D}(I \cap A)$$

which holds in any topological space, if I is open. We will also denote this modified base $\mathcal{B}_n$.

Further simplifications are possible if we restrict a GLP-space X to its subspace $\mathbf{D}_{n+1}(X)$, for a fixed n. This allows to obtain a neater characterization of topologies τ_n, for $n \geqslant 1$, in terms of τ_0.

3.1 D_n-reflection

The following notion seems rather useful in the study of **GLP**-spaces.

Definition 3.5 A set A $\mathbf{D}_n$*-reflects at* $x \in X$, if $x \in A$ implies $x \in \mathbf{D}_n(A)$. A point $x \in X$ is called $\mathbf{D}_n$*-reflexive* if $x \in \mathbf{D}_n(X)$ and, for each $A \subseteq X$, $\mathbf{D}_n(A)$ reflects at x. Notice that the converse inclusion $\mathbf{D}_n(\mathbf{D}_n(A)) \subseteq \mathbf{D}_n(A)$ is always true by the T_d property of τ_n.

Similarly, $x \in X$ is called *m-fold* $\mathbf{D}_n$*-reflexive* if $x \in \mathbf{D}_n(X)$ and each set of the form $\mathbf{D}_n(A_1) \cap \cdots \cap \mathbf{D}_n(A_m)$ reflects at x.

The next lemma shows that m-fold reflection, for each finite m, follows from just 2-fold reflection.

Lemma 3.6 *Every 2-fold reflexive point $x \in X$ is m-fold reflexive.*

Proof. Induction on $m \geqslant 2$, the argument works in any T_d space, so we omit the subscript n for readability. Suppose $x \in \mathbf{D}(A_1) \cap \cdots \cap \mathbf{D}(A_{m+1})$, then $x \in \mathbf{D}(A_1) \cap \cdots \cap \mathbf{D}(A_m)$ and $x \in \mathbf{D}(A_{m+1})$. By induction hypothesis,

$$x \in \mathbf{D}(\mathbf{D}(A_1) \cap \cdots \cap \mathbf{D}(A_m))$$

and by 2-fold reflection

$$x \in \mathbf{D}(\mathbf{D}(\mathbf{D}(A_1) \cap \cdots \cap \mathbf{D}(A_m)) \cap \mathbf{D}(A_{m+1})).$$

However, by T_d property

$$\mathbf{D}(\mathbf{D}(A_1) \cap \cdots \cap \mathbf{D}(A_m)) \subseteq \mathbf{D}(A_1) \cap \cdots \cap \mathbf{D}(A_m),$$

hence

$$x \in \mathbf{D}(\mathbf{D}(A_1) \cap \cdots \cap \mathbf{D}(A_m) \cap \mathbf{D}(A_{m+1})),$$

as required. $\boxtimes$

Remark 3.7 This argument has a well-known analogue in provability logic (first formulated by Japaridze). There $\mathbf{D}$-reflection corresponds to the local Σ_1-reflection principle, usually stated in a dual form.

In ordinal GLP-spaces $\mathbf{D}_1$-reflection corresponds to the so-called *stationary reflection*, well-studied in set theory. This will be explained below.

Proposition 3.8 *Points $x \in \mathbf{D}_{n+1}(X)$ are m-fold $\mathbf{D}_k$-reflexive, for each m and each $k \leqslant n$.*

Proof. We give an argument in modal logic format. Reasoning in **GLP**, it is sufficient to prove the formal statement of 2-fold reflection:

$$\langle n+1\rangle\top \wedge \langle k\rangle p \wedge \langle k\rangle q \to \langle n+1\rangle(\langle k\rangle p \wedge \langle k\rangle q).$$

Suppose the premise holds, then by Lemma 3.3 we obtain

$$\langle n+1\rangle\langle k\rangle p \wedge \langle k\rangle q,$$

and by the same lemma once again

$$\langle n+1\rangle(\langle k\rangle p \wedge \langle k\rangle q).$$

The latter formula can be weakened to

$$\langle k\rangle(\langle k\rangle p \wedge \langle k\rangle q)$$

by the monotonicity axiom of **GLP**, as required. ⊠

3.2 Decomposition

Suppose $(X;\tau_0,\tau_1,\dots)$ is a GLP-space generated from $(X;\tau_0)$. The following two propositions provide a neat characterization of the topologies τ_n, for $n \geqslant 1$, in terms of τ_0.

Proposition 3.9 *Topology τ_{n+1} of a GLP-space X restricted to the subspace $\mathbf{D}_{n+1}(X)$ has a base consisting of sets of the form*

$$A_0 \cap \mathbf{D}_n(A_1) \cap \mathbf{D}_{n+1}(X)$$

with $A_1 \subseteq A_0$, $A_0 \in B_0$.

Proof. Base B_{n+1} restricted to $\mathbf{D}_{n+1}(X)$ consists of sets of the form

$$A_0 \cap \mathbf{D}_k(A_1) \cap \dots \cap \mathbf{D}_k(A_m) \cap \mathbf{D}_{n+1}(X), \qquad (**)$$

where $k \leqslant n$, $A_0 \in B_0$. By Proposition 3.8

$$\mathbf{D}_k(A_1) \cap \dots \cap \mathbf{D}_k(A_m) \cap \mathbf{D}_{n+1}(X)$$

equals

$$\mathbf{D}_k(\mathbf{D}_k(A_1) \cap \dots \cap \mathbf{D}_k(A_m)) \cap \mathbf{D}_{n+1}(X).$$

The latter equals

$$\mathbf{D}_n(\mathbf{D}_k(\mathbf{D}_k(A_1) \cap \dots \cap \mathbf{D}_k(A_m))) \cap \mathbf{D}_{n+1}(X),$$

by the identity

$$\mathbf{D}_{n+1}(X) \cap \mathbf{D}_k(C) \subseteq \mathbf{D}_n(\mathbf{D}_k(C)),$$

proved similarly to Proposition 3.8. Hence, $(**)$ equals

$$A_0 \cap \mathbf{D}_n(A_0 \cap \mathbf{D}_k(\mathbf{D}_k(A_1) \cap \cdots \cap \mathbf{D}_k(A_m))) \cap \mathbf{D}_{n+1}(X),$$

which has the required format. ⊠

The following proposition characterizes $\mathbf{D}_{n+1}(X)$ in terms of τ_n.

Proposition 3.10 $\mathbf{D}_{n+1}(X)$ *coincides with the set of 2-fold* $\mathbf{D}_n$*-reflexive points.*

Proof. It is sufficient to show that each reflexive point belongs to $\mathbf{D}_{n+1}(X)$. Suppose x is $\mathbf{D}_n$-reflexive. Since $x \in \mathbf{D}_n(X)$, by Proposition 3.8 x is m-fold $\mathbf{D}_k$-reflexive, for each $k \leqslant n$. Consider any B_{n+1}-set

$$U := A_0 \cap \mathbf{D}_k(A_1) \cap \cdots \cap \mathbf{D}_k(A_m),$$

containing x. Since

$$x \in \mathbf{D}_k(A_1) \cap \cdots \cap \mathbf{D}_k(A_m),$$

by m-fold $\mathbf{D}_k$-reflexivity we obtain

$$x \in \mathbf{D}_k(\mathbf{D}_k(A_1) \cap \cdots \cap \mathbf{D}_k(A_m)).$$

Since A_0 is an open neighborhood of x, there is a $y \in A_0$ such that $y \neq x$ and

$$y \in \mathbf{D}_k(A_1) \cap \cdots \cap \mathbf{D}_k(A_m).$$

Hence $y \in U$ and $y \neq x$, as required. ⊠

Notice that every point of $X \setminus \mathbf{D}_{n+1}(X)$ is isolated, hence X is partitioned into $X \setminus \mathbf{D}_{n+1}(X)$ on which the topology is trivial, and the part $\mathbf{D}_{n+1}(X)$ on which there is a neat base for τ_{n+1}. This provides the following characterization of topology τ_{n+1} in terms of neighborhood bases.

Corollary 3.11 *A set A is τ_{n+1}-open iff, for each $x \in A$, if x is $\mathbf{D}_n$-reflexive, then there is a set $U := A_1 \cap \mathbf{D}_n(A_2)$ such that $x \in U \subseteq A$, $A_1 \in B_0$ and $A_2 \subseteq A_1$.*

4 Ordinal spaces

Here we shall characterize the topologies in ordinal GLP-spaces. It would be convenient for us to start with left order topology on an (unspecified and very big) ordinal Ω, rather than with the interval topology. It is the same topology as the upset topology on the inverse ordering Ω^*. We shall see that the usual interval topology will be obtained at the first step, so our study will naturally extend to ordinal spaces proper. For each of the generated topologies we establish a little vocabulary translating the general concepts from the previous sections.

4.1 Left topology, τ_0

1. *U is a neighborhood of* α: U contains $[0, \alpha]$.

2. *Isolated points*: $\{0\}$.

3. *Limit points* $\mathbf{D}(X)$: $(0, +\infty)$.

4. $\alpha \in \mathbf{D}(A)$: $\exists \beta < \alpha \, \beta \in A$.

5. $\mathbf{D}(A)$: $(\min(A), +\infty)$.

6. *A* $\mathbf{D}$*-reflects at* α: $\alpha \neq \min(A)$.

7. α *is* $\mathbf{D}$*-reflexive*: α is a limit ordinal.

   Indeed, reflexivity means $\alpha \neq 0$ and $\alpha \neq \min(\mathbf{D}(A))$, for each A, but $\min(\mathbf{D}(A)) = \min(A) + 1$ is always a successor ordinal.

8. α *is 2-fold reflexive*: α is a limit ordinal. Here, the notion coincides with the reflexivity.

9. *Base of open neighborhoods of* α *in the next topology*: $(\beta, \alpha]$, for $\alpha \in \mathrm{Lim}$, $\beta < \alpha$; $\{\alpha\}$ for $\alpha \notin \mathrm{Lim}$.

   This follows by Corollary 3.11, since

   $$[0, \alpha] \cap (\beta, +\infty) = (\beta, \alpha].$$

   Thus, the next topology is the familiar interval topology on κ.

4.2 Interval topology, τ_1

1. *U is a neighborhood of α*: U contains $(\beta, \alpha]$, for some $\beta < \alpha$.
2. *Isolated points*: 0 and successor ordinals.
3. *Limit points* $\mathbf{D}(X)$: limit ordinals Lim.
4. $\alpha \in \mathbf{D}(A)$: $\alpha \in \mathrm{Lim}$ and $A \cap \alpha$ is cofinal in α.
5. $\mathbf{D}(A)$: limit points of A.
6. *A $\mathbf{D}$-reflects at α*: If $\alpha \in A$ then $A \cap \alpha$ is cofinal in α.
7. *α is $\mathbf{D}$-reflexive*: $\mathrm{cf}(\alpha) > \omega$.

   Indeed, reflexivity of α means that $\alpha \in \mathrm{Lim}$ and, for all A, if A is cofinal in α, then $\mathbf{D}(A)$ is cofinal in α. If $\mathrm{cf}(\alpha) = \omega$ then there is an increasing sequence $(\alpha_n)_{n<\omega}$ such that $\alpha_n \to \alpha$. Then, for $A := \{\alpha_n : n < \omega\}$ we obviously have $\mathbf{D}(A) = \{\alpha\}$, so A violates the reflexivity property. If $\mathrm{cf}(\alpha) > \omega$ and A is cofinal in α, then $\mathbf{D}(A)$ is also cofinal in A. Suppose $\beta < \alpha$, consider the first ω-many elements B of A occurring above β. Let $\gamma := \sup B$. Obviously, A is cofinal in γ. Since $\mathrm{cf}(\alpha) > \omega$, we have $\gamma < \alpha$, q.e.d.
8. *α is 2-fold reflexive*: $\mathrm{cf}(\alpha) > \omega$.
9. *Base of open neighborhoods of α in the next topology*: $\mathbf{D}(A) \cap [0, \alpha]$, if $\mathrm{cf}(\alpha) > \omega$ and A is cofinal in α; $\{\alpha\}$ if not $\mathrm{cf}(\alpha) > \omega$.

   Notice that $\mathbf{D}(A) \cap \alpha$ is a club (closed unbounded set) in α, if A is cofinal in α. Also, for any club C in α, $\mathbf{D}(C) \subseteq C$, since C is closed. So, the filter of (pointed) neighborhoods of α with $\mathrm{cf}(\alpha) > \omega$ coincides with the so-called *club filter* in α. We call this topology the *club topology.*

4.3 Club topology, τ_2

1. *U is a neighborhood of α*: U contains α and, if $\mathrm{cf}(\alpha) > \omega$, U contains a club C in α.
2. *Isolated points*: $\{\alpha : \mathrm{cf}(\alpha) = \omega\}$, 0 and successor ordinals.
3. *Limit points* $\mathbf{D}(X)$: $\mathrm{cof}(> \omega) := \{\alpha : \mathrm{cf}(\alpha) > \omega\}$.

4. $\alpha \in \mathbf{D}(A)$: $A \cap \alpha$ is *stationary* in α.

   We say that $A \subseteq \alpha$ is stationary in α, if $\mathrm{cf}(\alpha) > \omega$ and A intersects every club in α (see [7, 6]).

5. $\mathbf{D}(A)$: $\{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}$.

   This is closely related to the so-called *Mahlo operation.* $M(A)$ is usually defined as $\{\alpha \in A : A \cap \alpha \text{ is stationary in } \alpha\}$. So,

   $$M(A) = \mathbf{D}(A) \cap A.$$

   Thus, if A is 2-closed, then $\mathbf{D}(A) = M(A)$.

Example 4.1 λ is a *weakly Mahlo cardinal* if $\{\rho < \lambda : \rho \text{ is regular}\}$ is stationary in λ. Since the class Reg of regular cardinals is 2-closed, we have: λ is weakly Mahlo iff $\lambda \in \mathbf{D}_2(\mathrm{Reg})$ iff $\lambda \in M(\mathrm{Reg})$.

6. *A $\mathbf{D}$-reflects at α*: If $\alpha \in A$ then $A \cap \alpha$ is stationary in α.

7. *α is $\mathbf{D}$-reflexive*: *Stationary reflection* holds in α.

   By this we mean: $\mathrm{cf}(\alpha) > \omega$ and, for all $A \subseteq \alpha$ stationary in α, there is a $\beta < \alpha$ such that $A \cap \beta$ is stationary in β (see [8]).

   D-reflexivity obviously implies stationary reflection. For the other direction, assume that A is stationary in α and notice that if C is a club in α and A is stationary in α, then $C \cap A$ is stationary. Reflecting $C \cap A$ below α for every club C delivers a collection of $\beta \in \mathbf{D}(A)$ intersecting every club C. Hence, $\mathbf{D}(A)$ is stationary in α.

8. *α is 2-fold reflexive*: *Simultaneous reflection* holds in α for pairs of stationary sets, that is, $\mathrm{cf}(\alpha) > \omega$ and for all A_1, A_2 stationary in α there is a $\beta < \alpha$ such that both $A_1 \cap \beta$ and $A_2 \cap \beta$ are stationary in β.

   This principle is known to be stronger than stationary reflection, at least under some restrictions (see below).

9. *Base of open neighborhoods of α in the next topology*:

   $$\{\beta \leqslant \alpha : A \cap \beta \text{ is stationary in } \beta\},$$

   if α is 2-fold reflexive and $A \cap \alpha$ is stationary in α; $\{\alpha\}$ if α is not 2-fold reflexive.

   We call this topology τ_3 *Mahlo topology*, because its open sets are defined using Mahlo operation.

5 Discussion

The first ordinal at which topology τ_n is not discrete is, obviously, $\delta_n := \min(\mathbf{D}_n(\kappa))$. We see that $\delta_0 = 1$, $\delta_1 = \omega$, $\delta_2 = \omega_1$. However, we do not know within ZFC what is δ_3 and whether such an ordinal even exists at all.

Stationary reflection has attracted considerable attention by set theorists with contributors such as Jech, Shelah, Magidor, Harrington, Solovay and many others (see [**?**] for an overview). Here we mention only the following results.

It is well known that every weakly compact cardinal is (2-fold) reflexive. Moreover, under the assumption $V = L$ only such cardinals are reflexive (Jensen [4]).

However, in general, δ_3 need not be very big. In fact, under some strong large cardinal assumptions one can force δ_3 to be equal $\aleph_{\omega+1}$ (Magidor, [8]). In general, we also know by simple arguments that reflexive ordinals have to be regular cardinals, but not successors of regular cardinals. In other words, reflexive ordinals are either weakly inaccessible or successors of singular cardinals (such as $\aleph_{\omega+1}$).

The consistency strength of the assertion that $\delta_3 = \aleph_{\omega+1}$ exists can be located between the existence of a measurable cardinal and that of infinitely many supercompact cardinals (Magidor [8], Dodd–Jensen [**?**]).

I do not know exactly, what is the consistency strength of the weaker assertion "δ_3 *exists.*" Does it imply the consistency of any large cardinal axiom? For example, does it imply that it is consistent that a weakly compact cardinal exists? (We obviously cannot do any better than that because weakly compact cardinals are reflexive.)

6 Non-triviality of the topologies τ_n

Here we give a sufficient condition, due to Philipp Schlicht, for all the topologies τ_n to be non-discrete. We show that, if there exists a Π^1_n-indescribable cardinal κ, then τ_{n+1} is not discrete.

Recall the definition of Q-indescribable cardinal, for a class of second-order formulas Q (see Kanamori [5]). We assume Q to contain at least the class of all first order formulas (denoted Π^1_0).

Definition 6.1 A set A is *Q-indescribable in κ* if, for all $R \subseteq V_\kappa$ and all sentences $\varphi \in Q$,

$$(V_\kappa, \in, R) \models \varphi \ \Rightarrow \ \exists \alpha < \kappa \, (\alpha \in A \text{ and } (V_\alpha, \in, R \cap V_\alpha) \models \varphi).$$

Here R is an unspecified (unary) relation symbol in the language.

Let $A \in \mathcal{F}_\kappa$ if $A \subseteq \kappa$ and $\kappa \setminus A$ is not Q-indescribable in κ. $\mathcal{F}_\kappa$ is called the *Q-indescribable filter over κ.*

κ is called *Q-indescribable* if κ is Q-indescribable in κ, i.e., equivalently, if $\mathcal{F}_\kappa$ is a proper filter.

In would be convenient to translate these definitions into topological terms. The Q-indescribable filter will be perceived as the filter of pointed neighborhoods of κ in a certain *Q-describable topology τ_Q*. The topology can then be more directly defined as follows.

Definition 6.2 For any sentence $\varphi \in Q$ and any $R \subseteq V_\kappa$, let $U_\kappa(\varphi, R)$ denote the set

$$\{\alpha \leqslant \kappa : (V_\alpha, \in, R \cap V_\alpha) \vDash \varphi\}.$$

The topology τ_Q is generated by the subbase consisting of sets $U_\kappa(\varphi, R)$, for all κ, $\varphi \in Q$, $R \subseteq V_\kappa$.

Example 6.3 Intervals $[0, \kappa]$, for each κ, are open: Consider $\varphi = \top$.

Example 6.4 Intervals $(\lambda, \kappa]$, for $\lambda < \kappa$, are open. In particular, $\{\kappa\}$ is a basic open set, if κ is 0 or a successor ordinal. Consider $R = \{\lambda\}$ and $\varphi = (\exists x\ x \in R)$. We have:

$$(V_\alpha, \in, R \cap V_\alpha) \vDash \varphi \iff \alpha > \lambda.$$

Lemma 6.5 *The sets $U_\kappa(\varphi, R)$ form a base for τ_Q, if Q is any of the classes Π^1_n, for $n \geqslant 0$.*

Proof. First, we remark that the following relationship is obvious for $\lambda \leqslant \kappa$:

$$U_\kappa(\varphi, R) \cap [0, \lambda] = U_\lambda(\varphi, R \cap V_\lambda). \tag{$*$}$$

Consider an intersection of two subbase sets:

$$U_{\kappa_1}(\varphi_1, R_1) \cap U_{\kappa_2}(\varphi_2, R_2),$$

such that w.l.o.g. $\kappa_1 \leqslant \kappa_2$. By $(*)$ we can replace $U_{\kappa_2}(\varphi_2, R_2)$ by $U_{\kappa_1}(\varphi_2, R_2 \cap V_{\kappa_1})$. Hence, we can assume $\kappa_1 = \kappa_2 =: \kappa$.

Furthermore, we can assume κ to be a limit ordinal; otherwise, if $\kappa = \lambda + n$, for $n < \omega$ and $\lambda \in \mathrm{Lim}$, we have

$$U_\kappa(\varphi, R) = U_\lambda(\varphi, R \cap V_\lambda) \cup F,$$

where F is a finite set of successor ordinals. Each successor ordinal, by the previous example, is an element of the base. Hence, it is sufficient to show that

$$U_\kappa(\varphi_1, R_1) \cap U_\kappa(\varphi_2, R_2) = U_\kappa(\varphi, R),$$

for some φ, $R \subseteq V_\kappa$, if κ is a limit ordinal.

Since κ is limit V_κ is closed under pairing. Hence, if $R_1, R_2 \subseteq V_\kappa$ we can form the product $R_1 \times R_2 \subseteq V_\kappa$. Then,

$$U_\kappa(\varphi_1, R_1) \cap U_\kappa(\varphi_2, R_2) = U_\kappa(\varphi_1(R_1) \wedge \varphi_2(R_2), R_1 \times R_2).$$

Since R_1 and R_2 can be recovered from $R = R_1 \times R_2$ by first order definable projection, this shows that the intersection belongs to the base. ⊠

Let $\mathbf{D}_Q$ denote the derived set operator for this topology. Then it is easy to see that $\kappa \in \mathbf{D}_Q(A)$ holds iff A is Q-indescribable in κ. Hence, κ is Q-indescribable iff $\kappa \in \mathbf{D}_Q(\mathrm{On})$ iff $\kappa \in \mathbf{D}_Q(\kappa)$.

A *weakly compact cardinal* can be defined as the Π^1_1-indescribable one. An exercise 6.12 in Kanamori [5] (due to Lévy) states the following basic fact:

Proposition 6.6 *If A is stationary in κ, then $\{\alpha < \kappa : A \cap \alpha$ is stationary in $\alpha\}$ belongs to the Π^1_1-indescribable filter.*

In topological terms this is equivalent to the following statement (which will be superseded by the next proposition):

Proposition 6.7 *The Mahlo topology τ_3 is contained in $\tau_{\Pi^1_1}$.*

Proof. The statement $\mathrm{Club}(C)$ expressing that C is a club (in On) is naturally expressed as a first order formula with a second order variable C. Likewise, the fact that the cofinality of the universe is uncountable can be expressed by the Π^1_1-sentence

$$\mathrm{Cof}_{>\omega} := \forall X\, (X \subseteq \mathrm{On} \wedge |X| \leqslant \omega \to \exists \beta \forall \delta \in X\ \delta < \beta).$$

Hence,

$$(V_\alpha, \in) \vDash \mathrm{Cof}_{>\omega} \iff \mathrm{cf}(\alpha) > \omega.$$

Then, $A \subseteq \alpha$ is stationary in α iff

$$(V_\alpha, \in, A) \vDash \mathrm{Cof}_{>\omega} \wedge \forall C\, (\mathrm{Club}(C) \to \exists \beta \in A \cap C).$$

Hence,

$$\{\alpha < \kappa : A \cap \alpha \text{ is stationary in } \alpha\}$$

equals

$$\{\alpha < \kappa : (V_\alpha, \in, A \cap \alpha) \vDash \psi\},$$

for the above formula $\psi \in \Pi^1_1$. Hence, if A is stationary in κ, $\{\alpha < \kappa : A \cap \alpha \text{ is stationary in } \alpha\}$ will be an open neighborhood of κ in $\tau_{\Pi^1_1}$. ⊠

Proposition 6.8 *For any $n \geqslant 0$, τ_{n+2} is contained in $\tau_{\Pi^1_n}$.*

Proof. We shall show that, for each n, there is a Π^1_n-formula $\varphi_{n+1}(R)$ such that

$$\kappa \in \mathbf{D}_{n+1}(A) \iff (V_\kappa, \in, A \cap \kappa) \vDash \varphi_{n+1}(A \cap \kappa). \qquad (**)$$

This implies that, for each $\kappa \in \mathbf{D}_{n+1}(A)$, the set $U_\kappa(\varphi_{n+1}, A \cap \kappa)$ is a $\tau_{\Pi^1_n}$-open subset of $\mathbf{D}_{n+1}(A)$ containing κ. Hence, each $\mathbf{D}_{n+1}(A)$ is $\tau_{\Pi^1_n}$-open. Since τ_{n+2} is generated over τ_{n+1} by the open sets of the form $\mathbf{D}_{n+1}(A)$ for various A, we have $\tau_{n+2} \subseteq \tau_{\Pi^1_n}$.

We prove $(**)$ by induction on n. For $n = 0$, notice that $\kappa \in \mathbf{D}_1(A)$ iff ($\kappa \in \mathrm{Lim}$ and $A \cap \kappa$ is unbounded in κ) iff

$$(V_\kappa, \in, A \cap \kappa) \vDash \forall \alpha \exists \beta \in A \ \alpha < \beta.$$

For the induction step, notice that by Proposition 3.11

$$\begin{aligned} \kappa \in \mathbf{D}_{n+1}(A) \quad \iff \quad & (\kappa \text{ is 2-fold } \mathbf{D}_n\text{-reflexive}) \wedge \\ & \forall Y \subseteq \kappa \, (\kappa \in \mathbf{D}_n(Y) \to \exists \alpha < \kappa \, (\alpha \in A \wedge \alpha \in \mathbf{D}_n(Y))). \end{aligned}$$

Using the induction hypothesis, for some $\varphi_n(R) \in \Pi^1_{n-1}$ we have

$$\alpha \in \mathbf{D}_n(A) \iff (V_\alpha, \in, A \cap \alpha) \vDash \varphi_n(A \cap \alpha).$$

Hence, the second line of the expression for $\kappa \in \mathbf{D}_{n+1}(A)$ is equivalent to

$$(V_\kappa, \in, A \cap \kappa) \vDash \forall Y \subseteq \mathrm{On} \, (\varphi_n(Y) \to \exists \alpha \, (\alpha \in A \wedge \varphi_n^{V_\alpha}(Y \cap \alpha))).$$

By the induction hypothesis, this formula is Π^1_n.

To treat the first line of the formula, we prove the following lemma.

Lemma 6.9 *κ is 2-fold $\mathbf{D}_n$-reflexive iff $\kappa \in \mathbf{D}_n(\mathrm{On})$ and*

$$\forall Y_1, Y_2 \subseteq \kappa \, (\kappa \in \mathbf{D}_n(Y_1) \cap \mathbf{D}_n(Y_2) \to \exists \alpha < \kappa \, \alpha \in \mathbf{D}_n(Y_1) \cap \mathbf{D}_n(Y_2)).$$

Proof. The 'only if' part is obvious. To prove the 'if' part, consider any $\kappa \in \mathbf{D}(Y_1) \cap \mathbf{D}_n(Y_2)$. We must show that, for each open neighborhood $U \in \mathcal{B}_n$ such that $U \ni \kappa$, there is a $\alpha \neq \kappa$ such that $\alpha \in U \cap \mathbf{D}_n(Y_1) \cap \mathbf{D}_n(Y_2)$.

If $U = A_0 \cap \mathbf{D}_k(A_1) \cap \cdots \cap \mathbf{D}_k(A_m)$, for some $k < n$ and $A_0 \in \mathcal{B}_0$, then

$$U \cap \mathbf{D}_n(Y_1) \cap \mathbf{D}_n(Y_2) = A_0 \cap \mathbf{D}_n(Y_1') \cap \mathbf{D}_n(Y_2),$$

where

$$Y_1' := Y_1 \cap \mathbf{D}_k(A_1) \cap \cdots \cap \mathbf{D}_k(A_m).$$

Hence, we obtain an $\alpha < \kappa$ such that $\alpha \in A_0 \cap \mathbf{D}_n(Y_1') \cap \mathbf{D}_n(Y_2)$. It follows that $\alpha \in U \cap \mathbf{D}_n(Y_1) \cap \mathbf{D}_n(Y_2)$, as required. ⊠

Similarly to the above, this formula can also be rewritten to a Π^1_n-format. ⊠

Corollary 6.10 *If there is a Π^1_n-indescribable cardinal, then τ_{n+2} has a non-isolated point.*

Corollary 6.11 *If there is a cardinal which is Π^1_n-indescribable, for each n, then all τ_n are non-trivial.*

7 Iterated derivatives

For a topological space X and a set $A \subseteq X$ define subsets $\mathbf{D}^\alpha[A]$ by transfinite recursion as follows.

$$\mathbf{D}^0[A] := X; \quad \mathbf{D}^{\alpha+1}[A] := \mathbf{D}(\mathbf{D}^\alpha[A] \cap A); \quad \mathbf{D}^\lambda[A] = \bigcap_{\alpha<\lambda} \mathbf{D}^\alpha[A].$$

We also define $D^\alpha := \mathbf{D}^\alpha[X]$.

If X is scattered, the sequence $\mathbf{D}^\alpha[A]$ is a strictly decreasing sequence of closed sets, hence we have $\mathbf{D}^\alpha[A] = \varnothing$, for some α. Rank of a scattered space X is the least α such that $D^\alpha = \varnothing$.

For GLP-space generated by the left topology on ordinals we characterize the iterations of its derived set operators $\mathbf{D}_0$, $\mathbf{D}_1$ and $\mathbf{D}_2$. $\mathbf{D}_n^\alpha[\mathrm{On}]$ will be denoted D_n^α. For each operator $\mathbf{D}_n$ we calculate the least fixed point ordinal, i.e., the least $\alpha > 0$ such that $\alpha \in D_n^\alpha$.

Proposition 7.1 $D_0^\alpha = [\alpha, +\infty)$.

Proof. By induction on α. ⊠

Proposition 7.2 *$D_1^\alpha = \{\omega^\alpha(1+\beta) : \beta \in \mathrm{On}\}$, if $\alpha > 0$.*

Proof. Basis: $D_1^1 = \mathrm{Lim} = \{\omega(1+\beta) : \beta \in \mathrm{On}\}$. Induction step:

$$D_1^{\alpha+1} = \mathbf{D}_1(D_1^\alpha) = \mathbf{D}_1(\{\omega^\alpha(1+\beta) : \beta \in \mathrm{On}\}).$$

However, $\mathbf{D}_1(A)$ is the set of limit points of A, hence

$$\begin{aligned}\mathbf{D}_1(\{\omega^\alpha(1+\beta) : \beta \in \mathrm{On}\}) &= \{\omega^\alpha(1+\beta) : \beta \in \mathrm{Lim}\} = \\ &= \{\omega^\alpha(\omega(1+\gamma)) : \gamma \in \mathrm{On}\} = \{\omega^{\alpha+1}(1+\gamma) : \gamma \in \mathrm{On}\}.\end{aligned}$$

Finally, if λ is a limit ordinal, we have

$$D_1^\lambda = \bigcap_{\alpha<\lambda} D_1^\alpha = \bigcap_{\alpha<\lambda} \{\omega^\alpha(1+\beta) : \beta \in \mathrm{On}\}.$$

We claim: $\delta \in \bigcap_{\alpha<\lambda}\{\omega^\alpha(1+\beta) : \beta \in \mathrm{On}\}$ iff $\delta = \omega^\lambda(1+\beta)$, for some β.

Suppose $\delta = \omega^\lambda(1+\beta)$. If $\alpha < \lambda$ then $\lambda = \alpha + \mu$, for some $\mu > 0$, hence $\omega^\lambda = \omega^\alpha\omega^\mu$ and $\delta = \omega^\alpha\omega^\mu(1+\beta) = \omega^\alpha(1+\beta')$, for some β'. Hence, $\delta \in \{\omega^\alpha(1+\beta) : \beta \in \mathrm{On}\}$.

Suppose $\delta = \omega^\lambda(1+\beta) + \gamma$, for some $\gamma < \omega^\lambda$. Then, by continuity, for some $\alpha < \lambda$ we have $\gamma < \omega^\alpha$. Then $\delta \notin \{\omega^\alpha(1+\beta) : \beta \in \mathrm{On}\}$. ⊠

Corollary 7.3 *$\mu\alpha > 0$. $\alpha \in D_1^\alpha = \varepsilon_0$.*

Proof. If $\alpha \in D_1^\alpha$, $\alpha > 0$, then $\alpha = \omega^\alpha(1+\beta)$ which implies $\beta = 0$ and $\omega^\alpha = \alpha$. ⊠

Let $\mathbf{r}_\alpha$ enumerate infinite regular cardinals: $\mathbf{r}_0 = \aleph_0$, $\mathbf{r}_1 = \aleph_1$, $\ldots$, $\mathbf{r}_\omega = \aleph_{\omega+1}$, etc.

Proposition 7.4 *$\beta \in D_2^\alpha \iff \mathrm{cf}(\beta) \geqslant \mathbf{r}_\alpha$, for $\alpha > 0$.*

Proof. Basis: $\beta \in D_2^1 \iff \mathrm{cf}(\beta) \geqslant \aleph_1$ is already shown.

Induction step: Let $\mathrm{cof}(\geqslant \lambda)$ denote the class of all ordinals α such that $\mathrm{cf}(\alpha) \geqslant \lambda$.

$$\begin{aligned}\beta \in D_2^{\alpha+1} &\iff (\beta \cap D_2^\alpha \text{ is stationary in } \beta \text{ and } \mathrm{cf}(\beta) > \omega) \\ &\iff (\mathrm{cf}(\beta) > \omega \text{ and } \mathrm{cof}(\geqslant \mathbf{r}_\alpha) \cap \beta \text{ is stationary in } \beta).\end{aligned}$$

We claim: $(\mathrm{cf}(\beta) > \omega$ and $\mathrm{cof}(\geqslant \mathbf{r}_\alpha) \cap \beta$ is stationary in $\beta)$ iff $\mathrm{cf}(\beta) \geqslant \mathbf{r}_{\alpha+1}$.

Suppose $\mathrm{cf}(\beta) \geqslant \mathbf{r}_{\alpha+1}$. Consider any club C in β. The order type of C is at least $\mathrm{cf}(\beta) \geqslant \mathbf{r}_{\alpha+1}$. Let $g(\gamma)$ denote the γ-th element of C and let $\lambda := g(\mathbf{r}_\alpha)$. We have $\mathrm{cf}(\lambda) = \mathbf{r}_\alpha$.

Indeed, $g : \mathbf{r}_\alpha \to \lambda$ is a cofinal map, hence $\mathrm{cf}(\lambda) \leqslant \mathbf{r}_\alpha$. If there also were a cofinal map $h : \mu \to \lambda$ with $\mu < \mathbf{r}_\alpha$, we would obtain a cofinal map $f : \mu \to \mathbf{r}_\alpha$ by setting

$$f(\gamma) := \min\{\nu : g(\nu) > h(\gamma)\}.$$

This contradicts the regularity of $\mathbf{r}_\alpha$. Hence, $\mathrm{cf}(\lambda) = \mathbf{r}_\alpha$ and $\mathrm{cof}(\geqslant \mathbf{r}_\alpha)$ intersects C.

Suppose $\mathrm{cf}(\beta) < \mathbf{r}_{\alpha+1}$. Let Y be a cofinal sequence in β of type $\lambda = \mathrm{cf}(\beta) \leqslant \mathbf{r}_\alpha$. Then the derived set $C := \mathbf{D}_1(Y)$ is a club in β. Moreover, for all $\gamma \in C$,

$$\mathrm{cf}(\gamma) \leqslant \mathrm{otyp}(C \cap \gamma) < \lambda \leqslant \mathbf{r}_\alpha.$$

Hence, $C \cap \mathrm{cof}(\geqslant \mathbf{r}_\alpha) \cap \beta = \varnothing$ and $\mathrm{cof}(\geqslant \mathbf{r}_\alpha) \cap \beta$ is not stationary in β.

For the limit ordinals λ, we have $\beta \in D_2^\lambda \iff \forall \alpha < \lambda\ \beta \in D_2^\alpha \iff \forall \alpha < \lambda\ \mathrm{cf}(\beta) \geqslant \mathbf{r}_\alpha \iff \mathrm{cf}(\beta) \geqslant \mathbf{r}_\lambda$. $\boxtimes$

Corollary 7.5 $\mu\alpha > 0$. $\alpha \in D_2^\alpha = \mu\alpha$. $(\alpha = \mathbf{r}_\alpha) = \lambda$, *where λ is the first weakly inaccessible cardinal.*

Proof. We prove that λ is weakly inaccessible iff $\lambda = \mathbf{r}_\lambda$.

If $\lambda = \mathbf{r}_\lambda$ then λ is a limit ordinal ($\forall \alpha\ \mathbf{r}_\alpha \in \mathrm{Lim}$). Hence, it is a limit of cardinals $\mathbf{r}_\alpha$, for $\alpha < \lambda$. It is also a regular cardinal, hence weakly inaccessible.

Suppose λ is a regular limit cardinal. Then $\mathrm{Reg} \cap \lambda$ is cofinal in λ, since for each $\alpha < \lambda$ we have $\alpha^+ < \lambda$ and α^+ is regular. Since λ is regular, $|\mathrm{Reg} \cap \lambda| = \lambda$. It follows that $\mathrm{Reg} \cap \lambda = \{\mathbf{r}_\beta : \beta < \lambda\}$, hence $\mathbf{r}_\lambda = \lambda$. $\boxtimes$

Corollary 7.6 (i) $D_2^1 \subseteq D_1^{\omega_1}$, *but* $D_2^1 \not\subseteq D_1^{\omega_1+1}$ *and* $D_1^{\omega_1+1} \not\subseteq D_2^1$;

(ii) $\mathbf{C}_1(D_2^1) = D_1^{\omega_1}$;

(iii) $\mathbf{D}_1(D_2^1) = D_1^{\omega_1+1}$.

Proof. (i) Let $\alpha \in D_2^1$, then $\mathrm{cf}(\alpha) \geqslant \omega_1$. Any α can be represented in the form $\alpha = \omega_1\beta + \gamma$ with $\gamma < \omega_1$. If $\mathrm{cf}(\alpha) \geqslant \omega_1$, we have $\gamma = 0$ and $\beta > 0$. Hence, by Proposition 7.2, $\alpha \in D_1^{\omega_1}$.

Similarly, we observe that $\omega_1\omega$ is the minimum of $D_1^{\omega_1+1}$, but $\mathrm{cf}(\omega_1\omega) = \omega$, hence $\omega_1\omega \notin D_2^1$. On the other hand, $\omega_1 \in D_2^1$, but $\omega_1 \notin D_1^{\omega_1+1}$.

(ii) Clearly, $\mathbf{C}_1(D_2^1) \subseteq D_1^{\omega_1}$, since $D_1^{\omega_1}$ is closed and (i) holds. If $\alpha \in D_1^{\omega_1}$ then $\alpha = \omega_1\beta$ with $\beta > 0$. If β is a successor, then $\mathrm{cf}(\alpha) = \omega_1$ hence $\alpha \in D_2^1$. If $\beta \in \mathrm{Lim}$ then β is a limit of successor ordinals. Hence $\omega_1\beta$ is a limit of ordinals of cofinality ω_1, that is, $\alpha \in \mathbf{C}_1(D_2^1)$.

(iii) This follows from (ii) as $\mathbf{D}_1\mathbf{C}_1 A = \mathbf{D}_1 A$, for any A. ⊠

To compare $\mathbf{D}_3$ and iterated $\mathbf{D}_2$ operators we prove the following proposition.

Proposition 7.7 *For any κ, $\kappa \in \mathbf{D}_3^1$ iff $\mathrm{cf}(\kappa) \in \mathbf{D}_3^1$.*

Proof. Let $\lambda = \mathrm{cf}(\kappa)$. We must show that λ is $\mathbf{D}_2$-reflexive iff so is κ. First, we need two basic auxiliary lemmas. They are well known and easy.

Lemma 7.8 *If $\lambda = \mathrm{cf}(\kappa)$ then there is a τ_1-continuous, strictly increasing and cofinal map $f : \lambda \to \kappa$.*

Proof. Since $\mathrm{cf}(\kappa) = \lambda$, there is a cofinal subset X in κ of order type λ. The set of limit points $Y := \mathbf{D}_1(X) \cap \kappa$ is a club in κ. In fact, Y in one-to-one correspondence with the limit ordinals below λ: if $\alpha < \lambda$ and $\alpha \in \mathrm{Lim}$, let $\alpha \mapsto \mathrm{otyp}(X \cap \alpha)$. Hence, $\mathrm{otyp}(Y) = \mathrm{otyp}(\lambda \cap \mathrm{Lim}) = \lambda$, since λ is a cardinal. The enumeration function for Y satisfies all the requirements. ⊠

Lemma 7.9 *Let $f : \lambda \to \kappa$ be as in the previous lemma. Then*

- *f is closed, i.e., $f''X$ is closed for every closed $X \subseteq \lambda$;*
- *X is a club in $\alpha \leqslant \lambda$ iff $f''X$ is a club in $f(\alpha) \leqslant \kappa$;*
- *X is stationary in $\alpha \leqslant \lambda$ iff $f''X$ is stationary in $f(\alpha) \leqslant \kappa$.*

Proof. (i) Assume $f''X \cap \alpha$ is cofinal in α. Let $X_\alpha := f^{-1}(f''X \cap \alpha)$ and let $\beta := \sup X_\alpha$. Since $X_\alpha \subseteq X$ and X is closed, we have $\beta \in X$. By continuity of f we obtain $f(\beta) = \sup f''X_\alpha = \sup f''X \cap \alpha = \alpha$.

(ii) If X is a club in α then $X \cup \{\alpha\}$ is closed, hence $f''X \cup \{f(\alpha)\}$ is closed, therefore $f''X$ is closed in $f(\alpha)$. Since $\alpha = \sup X$ by continuity we have $f(\alpha) = \sup f''X$, hence $f''X$ is unbounded in $f(\alpha)$.

If $f''X$ is a club in $f(\alpha)$, then $f''X \cup \{f(\alpha)\}$ is closed, hence $X \cup \{\alpha\} = f^{-1}(f''X \cup \{f(a)\})$ is closed. Hence, X is closed in α. Also, if $\sup X = \beta < \alpha$ then $\sup f''X = f(\beta) < f(\alpha)$. Hence, X is unbounded in α.

(iii) If X is stationary in α and C is a club in $f(\alpha)$, then $C \cap f''\alpha$ is a club in $f(\alpha)$. Hence, $C' := f^{-1}(C \cap f''\alpha)$ is a club in α (we have $f''C' = C \cap f''\alpha$).

Since X is stationary, $X \cap C' \neq \varnothing$. Hence, $f''X \cap C \cap f''\alpha = f''C' \neq \varnothing$, as required.

If $f''X$ is stationary in $f(\alpha)$ and C is a club in α, then $f''C$ is a club in $f(\alpha)$ and $f''C \cap f''X \neq \varnothing$. It follows that $C \cap X \neq \varnothing$. ⊠

Assume κ is reflexive for pairs of stationary sets and $\lambda = \mathrm{cf}(\kappa)$. Let $f : \lambda \to \kappa$ be a continuous increasing cofinal map. If A, B are stationary in λ, then $f''A, f''B$ are stationary in κ, hence there is a $\beta < \kappa$ such that $\beta \cap f''A, \beta \cap f''B$ are stationary in β. In particular, both of these sets are cofinal in β, and since $f''\lambda$ is closed $\beta \in f''\lambda$. Let $\alpha := f^{-1}(\beta)$. Since $f''(A \cap \alpha) = \beta \cap f''A$ and similarly for B, we obtain that $A \cap \alpha$ and $B \cap \alpha$ are stationary in α.

Assume $\lambda = \mathrm{cf}(\kappa)$ is reflexive for pairs of stationary sets. Let A, B be stationary in κ. Since $f''\lambda$ is a club in κ, $A \cap f''\lambda$ and $B \cap f''\lambda$ are stationary in κ. Let $A' := f^{-1}(A \cap f''\lambda)$ and $B' := f^{-1}(B \cap f''\lambda)$. A' and B' are stationary in λ, hence there is an $\alpha < \lambda$ such that $A' \cap \alpha$ and $B \cap \alpha$ are stationary in α. Then $f''(A' \cap \alpha)$ and $f''(B' \cap \alpha)$ are stationary in $f(\alpha) < \kappa$. Since $A \cap f(\alpha)$ contains $f''(A' \cap \alpha)$ and similarly for B, we obtain that $A \cap f(\alpha)$ and $B \cap f(\alpha)$ are stationary in $f(\alpha)$. ⊠

From this proposition we infer the following corollary. Let $\theta_3 := \min \mathbf{D}_3(\Omega)$ denote the first ordinal reflexive for pairs of stationary sets.

Corollary 7.10 *θ_3 is a regular cardinal.*

Proof. Indeed, since θ_3 is minimal, we have $\mathrm{cf}(\theta_3) = \theta_3$. ⊠

Proposition 7.11 (i) *$\kappa \in D_3^1$ implies $\mathrm{cf}(\kappa) \geqslant \theta_3$;*

(ii) *$\kappa \in \mathbf{C}_2(D_3^1)$ iff $\mathrm{cf}(\kappa) \geqslant \theta_3$;*

(iii) *$\mathbf{C}_2(D_3^1) = D_2^\mu$, where μ is such that $\theta_3 = \mathbf{r}_\mu$.*

Proof. (i) If $\kappa \in D_3^1$ then $\mathrm{cf}(\kappa) \in D_3^1$, hence $\mathrm{cf}(\kappa) \geqslant \theta_3$.

By Proposition 7.4, Statements (ii) and (iii) are equivalent. By (i), $D_3^1 \subseteq D_2^\mu$ and the latter is τ_2-closed. Hence $\mathbf{C}_2(D_3^1) \subseteq D_2^\mu$. We show $D_2^\mu \subseteq \mathbf{C}_2(D_3^1)$.

Assume $\mathrm{cf}(\kappa) \geqslant \theta_3$. If $\mathrm{cf}(\kappa) = \theta_3$ then $\kappa \in D_3^1$, by Proposition 7.7. If $\mathrm{cf}(\kappa) > \theta_3$ then the set

$$S := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \theta_3\}$$

is stationary in κ (Lemma 6.10 in Kunen [6]). By Proposition 7.7, this means $D_3^1 \cap \kappa$ is stationary in κ, hence $\kappa \in \mathbf{D}_2(D_3^1) \subseteq \mathbf{C}_2(D_3^1)$. ⊠

Remark 7.12 Assuming infinitely many supercompact cardinals, in some model of ZFC we have $\theta_3 = \aleph_{\omega+1}$. Then $\mu = \omega$ and we obtain $\mathbf{C}_2(D_3^1) = D_2^\omega$. However, it is also consistent with ZFC that θ_3 is a weakly compact cardinal (if such cardinals exist), and then μ must be the same cardinal.

8 Reduction property in GLP-spaces

An analog of the reduction property for GLP-spaces introduced below plays an important role in the proof-theoretic analysis of Peano arithmetic based on provability algebras (see [2]).

We begin with the following definitions. Let X be a scattered space.

Definition 8.1 A binary relation $<$ on $\mathcal{P}(X)$ is defined by

$$A < B \iff B \subseteq \mathbf{D}(A).$$

We also define $A \equiv B$ iff, for all $C \subseteq X$,

$$C < A \iff C < B.$$

Obviously, $<$ is transitive and $\equiv$ is an equivalence relation on $\mathcal{P}(X)$. If one excludes $\varnothing$, then $<$ is irreflexive, for $A \subseteq \mathbf{D}(A)$ implies A has no isolated points, whereas we assume X to be scattered. Hence, $<$ is a partial ordering relation on $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$.

If $(X; \tau_0, \tau_1, \dots)$ is a GLP-space, we denote $<$ and $\equiv$ for τ_n, respectively, $<_n$ and $\equiv_n$.

Example 8.2 For the left order topology τ_0 on Ω we have:

$$A <_0 B \iff B \subseteq (\min(A), +\infty) \iff \min(A) < \min(B).$$

Hence,

$$A \equiv_0 B \iff \min(A) = \min(B) \iff \mathbf{C}_0(A) = \mathbf{C}_0(B),$$

where $\mathbf{C}_0$ denotes the closure operator in τ_0. Notice that

- $<_0$ is well-founded on $\mathcal{P}^*(\Omega)$ and of height Ω;
- $A \equiv_0 B$ iff A and B are incomparable;
- $<_0$ linearly orders $\mathcal{P}(\Omega)/\!\equiv_0$.

Remark 8.3 The exact analogues of these notions in the provability GLP-algebra of an arithmetical theory T are: $A <_n B$ holds iff sentence B implies $\langle n \rangle_T A$, the n-consistency assertion for a sentence A. Relation $A \equiv_n B$ holds iff $T + A$ and $T + B$ prove the same sentences of the form $\langle n \rangle_T C$. This class of sentences is very close to the class of Π_{n+1}-sentences in arithmetic and, in fact, coincides with Π_{n+1} modulo equivalence in $T + \langle n \rangle_T \top$. Thus, $\equiv$ is essentially the notion of Π_{n+1}-conservativity in arithmetic (modulo $T + \langle n \rangle_T \top$).

We want to characterize $\equiv$ somewhat more generally. First, we observe the following properties.

Lemma 8.4 (i) *$A \equiv \mathbf{C}A$;*

(ii) *$\mathbf{C}A = \mathbf{C}B$ implies $A \equiv B$.*

Proof. Statement (i) follows from the fact that $\mathbf{D}C$ is closed, for each C. Statement (ii) follows from (i): if $\mathbf{C}A = \mathbf{C}B$ then $A \equiv \mathbf{C}A = \mathbf{C}B \equiv B$. ⊠

We would like to find out when the opposite implication in (ii) holds. Define:

$$\tilde{A} := \bigcap \{\mathbf{D}C : \mathbf{D}C \supseteq A\}.$$

Obviously, $\tilde{A}$ is a closed set containing A, hence $\mathbf{C}A \subseteq \tilde{A}$.

Lemma 8.5 (i) *$A \equiv \tilde{A}$;*

(ii) *$A \equiv B \iff \tilde{A} = \tilde{B}$.*

Proof. Statement (i) follows from the definition, and (ii) follows from (i). ⊠

In general, $\tilde{A}$ does not equal the closure of A. However, for a natural class of spaces this is so. Recall that a space X is *regular* (or T_3) if for any closed $A \subseteq X$ and any $a \notin A$ there are open $U \ni a$ and $V \supseteq A$ such that $U \cap V = \varnothing$.

Proposition 8.6 *If X is T_3 then $\tilde{A} = \mathbf{C}A$, if $A \subseteq \mathbf{D}(X)$, and $\tilde{A} = X$, otherwise.*

Proof. Assume $A \subseteq \mathbf{D}X$. We have to show $\tilde{A} \subseteq \mathbf{C}A$. Let $a \notin \mathbf{C}A$. Pick open $U \ni a$ and $V \supseteq \mathbf{C}A$ such that $U \cap V = \varnothing$. Letting $C := X \setminus U$ we prove $\mathbf{D}C \supseteq A$ and $a \notin \mathbf{D}C$.

Since C is closed, $\mathbf{D}C \subseteq C$, hence $a \notin \mathbf{D}C$.

Consider any $x \in A$ and any open neighborhood $U_x \ni x$. Since $V \cap U_x$ is open and $x \in A \subseteq \mathbf{D}X$, there is a point $y \neq x$ such that $y \in V \cap U_x$. Since $y \in V$ we also have $y \in C$. Hence, $x \in \mathbf{D}C$. ⊠

Corollary 8.7 *If X is T_3 then $A \equiv B$ holds iff $A, B \subseteq \mathbf{D}X$ and $\mathbf{C}A = \mathbf{C}B$, or both $A, B \nsubseteq \mathbf{D}X$.*

Proof. If $A \subseteq \mathbf{D}X$, then $\mathbf{C}A \subseteq \mathbf{D}X \neq X$. ⊠

Example 8.8 For the interval topology τ_1 on Ω we have: $A \equiv_1 B$ iff either $A, B \subseteq \mathrm{Lim}$ and $\mathbf{C}_1 A = \mathbf{C}_1 B$, or both $A, B \nsubseteq \mathrm{Lim}$. This follows from the fact that interval topology is T_3.

Let $A \subseteq \Omega$ be closed and $\alpha \notin A$. There is an interval $(\beta, \alpha]$ such that $A \cap (\beta, \alpha] = \varnothing$. Then let $V := [0, \beta] \cup (\alpha, +\infty)$ and $U := (\beta, \alpha]$.

Notice that the ordering $<_1$ is a subordering of $<_0$, hence it is also well-founded.

Proposition 8.6 also applies to all topologies τ_n of ordinal GLP-spaces, $n \geqslant 0$. Recall that a space X is *zero-dimensional*, if X has a base of clopen sets. The interval topology on Ω is zero-dimensional, since it has a base of intervals of the form $(\alpha, \beta]$ (for $\alpha < \beta$ including $\alpha = -\infty$, $\beta = +\infty$). The complement of $(\alpha, \beta]$ has the form $[0, \alpha] \cup (\beta, +\infty)$, which is a union of two basic open sets.

We obviously remark that if τ_0 is zero-dimensional then so are τ_n, for all n, in the generated GLP-space. Indeed, all base sets of the form $\mathbf{D}_n(A)$ will be open, and hence clopen, in τ_{n+1}. Thus, we obtain the following corollary.

Corollary 8.9 *In the ordinal GLP-space Ω, all the topologies τ_n are zero-dimensional.*

Lemma 8.10 *A zero-dimensional space is regular.*

Proof. If A is closed and $a \notin A$, then there is a basic clopen set U such that $a \in U$ and $U \cap A = \varnothing$. Then U and $X \setminus U$ play the role of U and V. ⊠

Corollary 8.11 *In the ordinal GLP-space Ω, for all $n \geqslant 1$, we have:*

$$A \equiv_n B \iff A, B \subseteq \mathbf{D}_n\Omega \text{ and } \mathbf{C}A = \mathbf{C}B, \text{ or both } A, B \nsubseteq \mathbf{D}_n\Omega.$$

Now we introduce a stronger version of equivalence on sets. It is an analog of the notion of *provable* Π_1-*conservativity.* In a provability algebra of a theory T this means that, for each sentence $\pi \in \Pi_1$,

$$\Box_T(A \to \pi) \leftrightarrow \Box_T(B \to \pi)$$

is true and provable in T. This can be written dually as

$$\Diamond_T(A \wedge \neg\pi) = \Diamond_T(B \wedge \neg\pi).$$

Restricting this to sentences π of the form $\Diamond_T C$ yields the following definition.

Definition 8.12 Define $A \cong B$ iff, for all $C \subseteq X$,

$$\mathbf{C}(A \cap -\mathbf{D}C) = \mathbf{C}(B \cap -\mathbf{D}C). \qquad (*)$$

In a GLP-space, we denote by $\cong_n$ the relation $\cong$ for topology τ_n.

Lemma 8.13 *If $A \cong B$ then $\mathbf{C}A = \mathbf{C}B$, $\mathbf{D}A = \mathbf{D}B$ and $A \equiv B$.*

Proof. Consider $C = \varnothing$ and use the identity $\mathbf{DC}A = \mathbf{D}A$. $\boxtimes$

Definition 8.14 A GLP-space X (and the corresponding dual algebra) satisfies (weak) *α-reduction property for $\mathbf{D}_n$* if, for each subset $A \subseteq X$,

$$\mathbf{D}_{n+1}(A) \equiv_n \mathbf{D}_n^\alpha[A].$$

X satisfies *strong* α-reduction property for $\mathbf{D}_n$ if, for each subset $A \subseteq X$,

$$\mathbf{D}_{n+1}(A) \cong_n \mathbf{D}_n^\alpha[A].$$

Remark 8.15 The provability algebra of elementary arithmetic satisfies the ω-reduction property for each n. This is the content of the reduction lemma [2].[1]

Obviously, the strong α-reduction property implies the weak one. For regular spaces we obtain the following characterization.

Lemma 8.16 *Suppose (X, τ_n) is regular. Then the weak reduction property for X is equivalent to the identity*

$$\mathbf{C}_n(\mathbf{D}_{n+1}A) = \mathbf{D}_n^\alpha[A],$$

for any $A \subseteq X$.

[1] Does it satisfy the strong reduction property?

Proof. Indeed, both $\mathbf{D}_{n+1}A$ and $\mathbf{D}_n^\alpha[A]$ are contained in $\mathbf{D}_n(X)$, hence Proposition 8.6 applies and

$$\mathbf{C}_n(\mathbf{D}_{n+1}A) = \mathbf{C}_n\mathbf{D}_n^\alpha[A].$$

However, $\mathbf{D}_n^\alpha[A]$ is closed, hence $\mathbf{C}_n\mathbf{D}_n^\alpha[A] = \mathbf{D}_n^\alpha[A]$. ⊠

Consider the GLP-space Ω generated by the left topology on ordinals.

Proposition 8.17 *Ω satisfies the strong ω-reduction property for $\mathbf{D}_0$.*

Proof. Recall that $\mathbf{D}_0(A) = (\min(A), +\infty)$ and hence $\mathbf{D}_0^+(A) = [\min(A), +\infty)$, the closure of A to the right. Similarly, $-\mathbf{D}_0(C) = [0, c]$ where $c = \min(C)$. $\mathbf{D}_1(A)$ is the set of limit points of A. $\mathbf{D}_0^\omega[A] = [a_\omega, +\infty)$ where a_ω is the ω-th element of A. For any set C we consider two cases.

Case 1. There is no limit point of A in the interval $[0, c]$. Then both sides of the equation $(*)$ are empty.

Case 2. There is a limit point of A in the interval $[0, c]$. Then a_ω belongs to $[0, c]$ and is the first limit point of A. Hence, both sides of the equation $(*)$ equal $[a_\omega, +\infty)$. ⊠

Proposition 8.18 *Ω satisfies the ω_1-reduction property for $\mathbf{D}_1$.*

Proof. This follows as in Lemma 7.6 (ii). ⊠

9 A upper bound result

Theorem 2 *Suppose A is a τ_0-closed subset of a GLB-space X. Then, for all ordinals $\alpha > 0$,*

$$\mathbf{D}_1^\alpha(A) \subseteq \mathbf{D}_0^{\omega^\alpha}(A).$$

For a proof of this theorem we need a few lemmas.

Lemma 9.1 (i) $\mathbf{D}_1(A) \cap \mathbf{D}_0(B) = \mathbf{D}_1(A \cap \mathbf{D}_0(B))$,

(ii) *For any $\alpha > 0$, $\mathbf{D}_1^\alpha(A) \cap \mathbf{D}_0(B) = \mathbf{D}_1^\alpha(A \cap \mathbf{D}_0(B))$.*

Proof. Part (i) is well-known to hold in any GLB-algebra. Part (ii) is proved by induction on α. Basis is Part (i). Induction step:

$$\mathbf{D}_1^{\alpha+1}A \cap \mathbf{D}_0B = \mathbf{D}_1(\mathbf{D}_1^\alpha A) \cap \mathbf{D}_0B = \mathbf{D}_1(\mathbf{D}_1^\alpha A \cap \mathbf{D}_0B) = \mathbf{D}_1\mathbf{D}_1^\alpha(A \cap \mathbf{D}_0B).$$

If λ is a limit, then

$$\mathbf{D}_1^\lambda A \cap \mathbf{D}_0 B = \mathbf{D}_0 B \cap \bigcap_{\alpha<\lambda} \mathbf{D}_1^\alpha A = \bigcap_{\alpha<\lambda} (\mathbf{D}_0 B \cap \mathbf{D}_1^\alpha A) = \bigcap_{\alpha<\lambda} \mathbf{D}_1^\alpha (A \cap \mathbf{D}_0 B) = \mathbf{D}_1^\lambda (A \cap \mathbf{D}_0 B).$$

$\boxtimes$

For a topological space X and a set $A \subseteq X$ define subsets $\mathbf{D}^\alpha[A]$ by transfinite recursion as follows.

$$\mathbf{D}^0[A] := X; \quad \mathbf{D}^{\alpha+1}[A] := \mathbf{D}(\mathbf{D}^\alpha[A] \cap A); \quad \mathbf{D}^\lambda[A] = \bigcap_{\alpha<\lambda} \mathbf{D}^\alpha[A].$$

Notice that $\mathbf{D}^\alpha[A]$ is a decreasing sequence of closed subsets of X. Moreover, if A is closed, then $\mathbf{D}^\alpha[A] = \mathbf{D}^\alpha A$.

Lemma 9.2 *For any $A \subseteq X$, $\mathbf{D}_1(A) \subseteq \mathbf{D}_0^\omega[A]$.*

Proof. We show by induction on $n < \omega$ that $\mathbf{D}_1(A) \subseteq \mathbf{D}_0^n[A]$. The claim is obvious for $n = 0$. Suppose the claim holds for $n = k$. Then

$$\mathbf{D}_1(A) \subseteq \mathbf{D}_1(A) \cap \mathbf{D}_0^k[A] = \mathbf{D}_1(A \cap \mathbf{D}_0^k[A]) \subseteq \mathbf{D}_0(A \cap \mathbf{D}_0^k[A]) = \mathbf{D}_0^{k+1}[A].$$

The first equality holds since $\mathbf{D}_0^k[A]$ has the form $\mathbf{D}_0(B)$. $\boxtimes$

Proof of Theorem. We argue by induction on α (with a quantifier over all closed sets A).

Basis: $\alpha = 1$. We must show $\mathbf{D}_1(A) \subseteq \mathbf{D}_0^\omega(A)$, that is, for all $n < \omega$, $\mathbf{D}_1(A) \subseteq \mathbf{D}_0^n(A)$. We prove it by a subsidiary induction on n. For $n = 0$ the claim is obvious, since A is closed. For $n = k+1$ we obtain:

$$\mathbf{D}_1(A) \subseteq \mathbf{D}_1(A) \cap \mathbf{D}_0^k(A) = \mathbf{D}_1(A \cap \mathbf{D}_0^k(A)) = \mathbf{D}_1 \mathbf{D}_0^k(A) \subseteq \mathbf{D}_0^{k+1}(A).$$

For the first equality we used Lemma 9.1 (i). For the second we used $A \supseteq \mathbf{D}_0^k(A)$, which is valid since A is closed.

Induction step. Suppose α is a limit ordinal. Then

$$\mathbf{D}_1^\alpha(A) = \bigcap_{\beta<\alpha} \mathbf{D}_1^\beta(A) \subseteq \bigcap_{\beta<\alpha} \mathbf{D}_0^{\omega^\beta}(A) = \mathbf{D}_0^{\omega^\alpha}(A).$$

The last equality follows from the fact that the sequence of sets of the form $\mathbf{D}_0^\beta(A)$ is decreasing and $\omega^\alpha = \sup_{\beta<\alpha} \omega^\beta$ if α is a limit ordinal.

Suppose $\alpha = \beta + 1$. By Lemma 9.2,

$$\mathbf{D}_1^{\beta+1}(A) = \mathbf{D}_1(\mathbf{D}_1^\beta A) \subseteq \mathbf{D}_0^\omega[\mathbf{D}_1^\beta A] = \bigcap_{n<\omega} \mathbf{D}_0^n[\mathbf{D}_1^\beta A].$$

We prove by induction on $n < \omega$ that

$$\mathbf{D}_0^n[\mathbf{D}_0^\beta A] \subseteq \mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot n} A. \qquad (*)$$

It will then follow that

$$\mathbf{D}_1^{\beta+1}(A) \subseteq \bigcap_{n<\omega} \mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot n} A = \mathbf{D}_0^{\omega^{\beta+1}} A.$$

If $n = 1$ the claim $(*)$ amounts to the induction hypothesis for β.

If $(*)$ holds for $n = k$, then we obtain:

$$\begin{aligned} \mathbf{D}_0^{n+1}[\mathbf{D}_1^\beta A] = \mathbf{D}_0(\mathbf{D}_1^\beta A \cap \mathbf{D}_0^n[\mathbf{D}_1^\beta A]) \subseteq \mathbf{D}_0(\mathbf{D}_1^\beta A \cap \mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot n} A) = \\ = \mathbf{D}_0(\mathbf{D}_1^\beta (A \cap \mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot n} A)) = \mathbf{D}_0 \mathbf{D}_1^\beta \mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot n} A \quad (1) \end{aligned}$$

By the induction hypothesis applied to the closed set $\mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot n} A$ in place of A we obtain

$$\mathbf{D}_0 \mathbf{D}_1^\beta \mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot n} A \subseteq \mathbf{D}_0(\mathbf{D}_0^{\omega^\beta} \mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot n} A) = \mathbf{D}_0 \mathbf{D}_0^{\omega^\beta \cdot (n+1)} A.$$

Here we also used the obvious fact that $\mathbf{D}^\alpha \mathbf{D}^\beta B = \mathbf{D}^{\beta+\alpha} B$, for any α, β, B. $\boxtimes$

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