Modal propositional inference rules for PA

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An inference rule A/B is said to be admissible in PA if for every arithmetical realization f, $\mathsf{PA} \vdash f(A)$ implies $\mathsf{PA} \vdash f(B)$. From the Solovay's second completeness theorem we immediately conclude

Theorem 1 *A/B is admissible in PA iff $\mathbf{S} \vdash \Box A \to \Box B$.*

Corollary 1 *Admissibility of a rule in PA is a decidable property.*

Here we provide another characterization of admissible rules that gives a left adjoint. Write $A \vdash B$ iff B is provable from the axioms of **GL** and A using modus ponens and necessitation rules.

Theorem 2 *For any formula A one can effectively construct a formula A^* such that for all φ,*

$$A/\varphi \text{ is admissible in } \mathsf{PA} \iff A^* \vdash \varphi.$$

We begin with a notion of Parikh provability. *Parikh rule* is the PA- and **GL**-admissible rule $\Box\varphi/\varphi$. We write $A \vdash_P B$ iff B is provable from the axioms of **GL** and A using modus ponens, necessitation and Parikh rules.

First, we obtain a useful characterization.

Theorem 3 *The following statements are equivalent:*

(i) $A \vdash_P \varphi$;

(ii) $\mathbf{S} \vdash \Box A \to \Box\varphi$;

(iii) $A \vdash \Box^{n+1}\varphi$, *where $n := d(A)$ is the number of different $\Box$-subformulas of A;*

(iv) $A \vdash \Box^n \varphi$*, for some* n*.*

Proof. (i)⇒(ii) Induction on the length of the derivation. Obvious for all the axioms and inference rules.

(ii)⇒(iii) Assume $A \nvdash \Box^{n+1}\varphi$. Consider a Kripke model $\mathcal{K}$ such that $\mathcal{K} \Vdash \boxdot A$ and $\mathcal{K} \nVdash \Box^{n+1}\varphi$. ¿From the second condition we find in $\mathcal{K}$ a linear chain of nodes of length $n+1$ below a node, where φ is false. By the choice of n there must be an A-reflexive point r among them. Let $\mathcal{K}_r$ be the restriction of $\mathcal{K}$ to this node. Then $\mathcal{K}_r \Vdash \boxdot A$ and $\mathcal{K}_r \nVdash \Box\varphi$. Pulling a tail out of the root of $\mathcal{K}_r$ delivers a tail model where $\Box A$ and $\neg\Box\varphi$ are true.

(iii)⇒(iv) and (iv)⇒(i) are obvious. ⊠

Notice that this theorem implies that a rule is PA-admissible iff it is derivable in **GL** enriched with the Parikh rule. Thus, Parikh rule is, in a sense, the most general PA-admissible (propositional modal) rule.

Lemma 2 *For any formula* A *there is a formula* A' *such that, for all* φ*,*

$$\mathbf{GL} \vdash A \to \Box\varphi \iff \mathbf{GL} \vdash A' \to \varphi.$$

Proof. Write A as $\bigvee_i A_i$, where A_i has the form

$$\textstyle\bigwedge_j \Box\psi_{ij} \wedge \bigwedge_k \neg\Box\theta_{ik} \wedge \bigwedge_l p_{il} \wedge \bigwedge_m \neg p_{im}.$$

We may assume that $\mathbf{GL} \nvdash \neg A_i$, for each i, for inconsistent formulas A_i can be deleted from the disjunction.

Obviously, $\mathbf{GL} \vdash A \to \Box\varphi$ iff $\mathbf{GL} \vdash A_i \to \Box\varphi$, for all i. Consider any particular A_i. We claim:

$$\mathbf{GL} \vdash A_i \to \Box\varphi \iff \mathbf{GL} \vdash \bigwedge_j \boxdot\psi_{ij} \to \varphi.$$

Indeed, the implication ($\Leftarrow$) is obvious, by necessiation and the axioms of **K4**.

For the opposite implication, consider a Kripke model $\mathcal{K}_1$ such that $\mathcal{K}_1 \nVdash \varphi$ and $\mathcal{K}_1 \Vdash \boxdot\psi_{ij}$, for all j. Further, since we assumed A_i to be consistent, consider any model $\mathcal{K}_2$ of A_i. Attach the model $\mathcal{K}_1$ immediately above the root of $\mathcal{K}_2$. It is easy to see that the resulting model is a countermodel to $A_i \to \Box\varphi$.

Now we can define $A' := \bigvee_i \bigwedge_j \boxdot\psi_{ij}$. ⊠

As a corollary of this lemma we obtain a similar statement for provability in **GL** from assumptions.

Lemma 3 *For every A there is a formula A° such that, for all φ,*

$$A \vdash \Box\varphi \iff A^\circ \vdash \varphi.$$

Proof. We have

$$A \vdash \Box\varphi \iff \mathbf{GL} \vdash \boxdot A \to \Box\varphi \iff \mathbf{GL} \vdash (\boxdot A)' \to \varphi. \qquad (*)$$

We let $A^\circ := (\boxdot A)'$. The statement then follows from the fact that

$$\mathbf{GL} \vdash (\boxdot A)' \leftrightarrow \boxdot(\boxdot A)'.$$

This is easy to see substituting in $(*)$ the formula $(\boxdot A)'$ for φ. Indeed, $A \vdash \Box(\boxdot A)'$, hence $A \vdash \Box\Box(\boxdot A)'$, and so $\mathbf{GL} \vdash (\boxdot A)' \to \Box(\boxdot A)'$. ⊠

Proof of the Theorem. Let $A_0 := A$ and $A_{k+1} := A_k^\circ$. We let $A^* := A_{n+1}$, where $n = d(A)$. By an obvious induction on k, from the previous lemma, we prove:

$$A \vdash \Box^k\varphi \iff A_k \vdash \varphi.$$

Induction step:

$$A \vdash \Box^k\Box\varphi \iff A_k \vdash \Box\varphi \iff A_k^\circ \vdash \varphi.$$

So, from Theorem 3 we conclude that $A \vdash_P \varphi$ iff $A^* \vdash \varphi$. ⊠