Exceptional collections for Grassmannians of isotropic lines

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To Miles Reid on the occasion of his 60th birthday

Abstract
We construct a full exceptional collection of vector bundles in the derived categories of coherent sheaves on the Grassmannian of isotropic two-dimensional subspaces in a symplectic vector space of dimension $2n$ and in an orthogonal vector space of dimension $2n + 1$ for all $n$.

1. Introduction
The derived category of coherent sheaves is the most important algebraic invariant of an algebraic variety. This is one reason to investigate its structure. In general, the structure of a triangulated category (the derived category is triangulated) is quite complicated. However, there is an important case when it can be described fairly explicitly.

Definition 1.1 [2, 5]. A collection of objects $(E_1, E_2, \ldots, E_n)$ of a $k$-linear triangulated category $T$ is exceptional if

$$\text{RHom}(E_i, E_i) = k \quad \text{for all } i, \quad \text{and} \quad \text{RHom}(E_i, E_j) = 0 \quad \text{for all } i > j.$$ 

A collection $(E_1, E_2, \ldots, E_n)$ is full if the minimal triangulated subcategory of $T$ containing $E_1, E_2, \ldots, E_n$ coincides with $T$.

Triangulated categories possessing a full exceptional collection are the simplest among the others. Every object of a triangulated category $T$ with a full exceptional collection $(E_1, E_2, \ldots, E_n)$ admits a unique (functorial) filtration with $i$th quotient being a direct sum of shifts of $E_i$. Therefore, an exceptional collection can be considered as a kind of basis for the triangulated category. However, the condition of existence of a full exceptional collection in a triangulated category is quite restrictive. For example, a necessary (but not sufficient) condition for the derived category of coherent sheaves on a smooth projective variety $X$ to have a full exceptional collection is the vanishing of non-diagonal Hodge numbers; that is, $h^{ij}(X) = 0$ for $i \neq j$ (an example of a smooth projective variety with vanishing non-diagonal Hodge numbers not admitting a full exceptional collection is provided by Enriques surface).

The simplest example of a variety with a full exceptional collection is a projective space. Beilinson [1] in 1978 showed that the collection of line bundles $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ on $\mathbb{P}^n$ is a full exceptional collection. In 1988 Kapranov [6] constructed full exceptional collections on
Grassmannians and flag varieties of groups $SL_n$ and on smooth quadrics. It has been conjectured afterwards that:

**Conjecture 1.2.** Any projective homogeneous space of a semisimple algebraic group admits a full exceptional collection consisting of vector bundles.

**Conjecture 1.3.** The (complete) flag variety of a semisimple algebraic group admits a full exceptional collection consisting of line bundles.

It is somehow surprising that only a very little progress in this direction has been achieved. As it was already mentioned, Kapranov showed that both Conjectures 1.2 and 1.3 are true for the group $SL_n$. Also, it is easy to see that Conjecture 1.3 is true for the group $SP_{2n}$ (see [12]) since the corresponding flag variety can be represented as an iterated projectivization of vector bundles, and by [10] the derived category of a projectivization of a vector bundle admits a full exceptional collection consisting of line bundles if the base does.

If one wishes to establish Conjecture 1.2, that is, to construct a full exceptional collection on any homogeneous space, then it is natural to consider first the case of Grassmannians, that is, of the homogeneous varieties $G/P$, where $P$ is a maximal parabolic subgroup in a semisimple algebraic group $G$. Presumably, this would suffice to prove Conjecture 1.2, in general, via the parabolic induction procedure. Now let us briefly describe what is known about derived categories of Grassmannians of classical semisimple algebraic groups.

The group $(SL_n)$: The Grassmannians of the group $SL_n$ are the usual Grassmannians $Gr(k, n)$ of $k$-dimensional subspaces in an $n$-dimensional vector space for $1 \leq k \leq n - 1$. Let $U$ denote the tautological rank $k$ sub-bundle in the trivial vector bundle $O^{\oplus n}_{Gr(k, n)}$. Consider the corresponding principal $GL_k$-bundle on $Gr(k, n)$. Given a non-increasing collection of $k$ integers $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k)$, we consider it as a dominant weight of the group $GL_k$ and denote by $\Sigma^\alpha U$ the associated vector bundle on $Gr(k, n)$. In particular,

$$\text{for } \alpha = (m, 0, \ldots, 0), \quad \Sigma^\alpha U = S^m U \text{ is the } m\text{th symmetric power},$$

and

$$\text{for } \alpha = (1, \ldots, 1, 0, \ldots, 0), \quad \Sigma^\alpha U = \Lambda^m U \text{ is the } m\text{th exterior power}.$$
$OGr(m, 2m)$, and so it is not a Grassmannian. For these Grassmannians exceptional collections are known only in the following cases.

- $k = 1$ and odd $n$. In this case $OGr(1, n) \cong Q^{n-2}$ is an odd-dimensional quadric and $\{O_Q, S, O_Q(1), O_Q(2), \ldots, O_Q(n-3)\}$ is a full exceptional collection (see [6]) ($S$ is the spinor bundle).

- $k = 1$ and even $n$. In this case $OGr(1, n) \cong Q^{n-2}$ is an even-dimensional quadric and $\{O_Q, S^+, S^-, O_Q(1), O_Q(2), \ldots, O_Q(n-3)\}$ is a full exceptional collection (see [6]) ($S^\pm$ are the spinor bundles).

- $n \leq 6$. The group $SO_n$ in this case has a simpler description: $SO_3$ up to a finite subgroup coincides with $SL_2$; $SO_4$ up to a finite subgroup coincides with $SL_2 \times SL_2$; $SO_5$ up to a finite subgroup coincides with $SP_4$; and $SO_6$ up to a finite subgroup coincides with $SL_4$.

In the present paper we construct a full exceptional collection on $OGr(2, 2n + 1)$ for all $n$.

As we already mentioned above, the main result of the present paper is a construction of full exceptional collections on Grassmannians $SGr(2, 2m)$ and $OGr(2, 2m + 1)$. Let us look at the method of constructing and proving the fullness of these collections. First of all, we consider a special exceptional collection on $Gr(2, 2m)$:

$\begin{pmatrix}
S^{m-1}U^* & S^{m-1}U^*(1) & \cdots & S^{m-1}U^*(m-1) \\
S^{m-2}U^* & S^{m-2}U^*(1) & \cdots & S^{m-2}U^*(m-1) & S^{m-2}U^*(m) & \cdots & S^{m-2}U^*(2m-1) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
S^2U^* & S^2U^*(1) & \cdots & S^2U^*(m-1) & S^2U^*(m) & \cdots & S^2U^*(2m-1) \\
U^* & U^*(1) & \cdots & U^*(m-1) & U^*(m) & \cdots & U^*(2m-1) \\
O & O(1) & \cdots & O(m-1) & O(m) & \cdots & O(2m-1)
\end{pmatrix}$

(1)

(the ordering is from bottom to top in columns, the left column goes first). Note that this is not Kapranov’s collection. In comparison with Kapranov’s collection, (1) is more symmetric with respect to the $O(1)$-twisting. This symmetry property is axiomatized in the notion of a minimal Lefschetz exceptional collection.

A Lefschetz exceptional collection is just an exceptional collection which consists of several blocks, each of them is a sub-block of the previous one twisted by $O(1)$ (in the collection (1) the blocks are the columns). A Lefschetz exceptional collection is minimal if, roughly speaking, its first block is the minimal possible (see the precise definition in [7]). Kapranov’s collection on $Gr(2, 2m)$ is a Lefschetz collection with the first block $(O, U^*, S^2U^*, \ldots, S^{2m-2}U^*)$; however, it is not minimal, since the first block of the collection (1) is strictly smaller.

Lefschetz exceptional collections have many properties, for example, they behave well with respect to the restriction to a hyperplane section. Explicitly, if we remove the first block of a Lefschetz exceptional collection and then restrict the rest to a hyperplane, then we obtain a Lefschetz exceptional collection (see Proposition 2.4). Evidently, the smaller the first block, the bigger the collection we obtain on the hyperplane. In particular, the biggest exceptional collection on a hyperplane can be obtained from a minimal Lefschetz collection on the ambient variety.

Since $SGr(2, 2m)$ is a hyperplane section of $Gr(2, 2m)$, we obtain in this way a Lefschetz exceptional collection for $SGr(2, 2m)$:

$\begin{pmatrix}
S^{m-1}U^* & \cdots & S^{m-1}U^*(m-2) \\
S^{m-2}U^* & \cdots & S^{m-2}U^*(m-2) & S^{m-2}U^*(m-1) & \cdots & S^{m-2}U^*(2m-2) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
S^2U^* & \cdots & S^2U^*(m-2) & S^2U^*(m-1) & \cdots & S^2U^*(2m-2) \\
U^* & \cdots & U^*(m-2) & U^*(m-1) & \cdots & U^*(2m-2) \\
O & \cdots & O(m-2) & O(m-1) & \cdots & O(2m-2)
\end{pmatrix}$

(2)
(we applied to the obtained collection on $\text{SGr}(2, 2m)$ an additional $O(-1)$-twist for convenience). This collection turns out to be full. To prove this we use the fact that this exceptional collection behaves well with respect to the restriction to any $\text{SGr}(2, 2m - 2) \subset \text{SGr}(2, 2m)$, and use induction on $m$.

Similarly, in the case of the orthogonal Grassmannian $O\text{Gr}(2, 2m + 1)$ we start with a Lefschetz exceptional collection on $\text{Gr}(2, 2m + 1)$:

$$
\begin{pmatrix}
S^{m-1}U^* & S^{m-1}U^*(1) & \cdots & S^{m-1}U^*(m-1) & S^{m-1}U^*(m) & \cdots & S^{m-1}U^*(2m) \\
S^{m-2}U^* & S^{m-2}U^*(1) & \cdots & S^{m-2}U^*(m-1) & S^{m-2}U^*(m) & \cdots & S^{m-2}U^*(2m) \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
S^2U^* & S^2U^*(1) & \cdots & S^2U^*(m-1) & S^2U^*(m) & \cdots & S^2U^*(2m) \\
U^* & U^*(1) & \cdots & U^*(m-1) & U^*(m) & \cdots & U^*(2m) \\
O & O(1) & \cdots & O(m-1) & O(m) & \cdots & O(2m)
\end{pmatrix}
$$

This time $O\text{Gr}(2, 2m + 1) \subset \text{Gr}(2, 2m + 1)$ is not a hyperplane section, rather it is a section of a vector bundle $S^2U^*$. Nevertheless, a subcollection of (3) restricts to a Lefschetz exceptional collection on $O\text{Gr}(2, 2m + 1)$. However, in a contrast with the symplectic case, it is not full. Similarly, as in the case of quadrics we need suitably defined spinor bundles on $O\text{Gr}(2, 2m + 1)$. These bundles can be defined as follows.

Consider the partial flag variety $O\text{F}(2, m; 2m + 1) \subset O\text{Gr}(2, 2m + 1) \times O\text{Gr}(m, 2m + 1)$, and denote by $\pi_2 : O\text{F}(2, m; 2m + 1) \rightarrow O\text{Gr}(2, 2m + 1)$, $\pi_m : O\text{F}(2, m; 2m + 1) \rightarrow O\text{Gr}(m, 2m + 1)$ the projections. We define the spinor bundle $S$ on $O\text{Gr}(2, 2m + 1)$ as

$$
S = \pi_2 \pi_m^* \mathcal{O}_{O\text{Gr}(m, 2m+1)}(1),
$$

where $\mathcal{O}_{O\text{Gr}(m, 2m+1)}(1)$ is the positive generator of $\text{Pic}(O\text{Gr}(m, 2m+1))$ (which is not the pullback of $\mathcal{O}_{O\text{Gr}(2m+1,2m+1)}(1)$; actually the pullback of $\mathcal{O}_{O\text{Gr}(2m+1,2m+1)}(1)$ is isomorphic to $\mathcal{O}_{O\text{Gr}(m, 2m+1)}(2)$). It turns out that the twists of spinor bundles can be inserted in the exceptional collection such that the collection

$$
\begin{pmatrix}
S & S(1) & \cdots & S(m-1) & S(m) & \cdots & S(2m-3) \\
S^{m-2}U^* & S^{m-2}U^*(1) & \cdots & S^{m-2}U^*(m-1) & S^{m-2}U^*(m) & \cdots & S^{m-2}U^*(2m-3) \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
S^2U^* & S^2U^*(1) & \cdots & S^2U^*(m-1) & S^2U^*(m) & \cdots & S^2U^*(2m-3) \\
U^* & U^*(1) & \cdots & U^*(m-1) & U^*(m) & \cdots & U^*(2m-3) \\
O & O(1) & \cdots & O(m-1) & O(m) & \cdots & O(2m)
\end{pmatrix}
$$

is exceptional. To prove its fullness we use the fact that this exceptional collection behaves well with respect to the restriction to any $O\text{Gr}(2, 2m - 1) \subset O\text{Gr}(2, 2m + 1)$, and use induction on $m$.

It is also worth mentioning that the notion of a Lefschetz exceptional collection (or more generally of a Lefschetz semi-orthogonal decomposition) was introduced in [9] as a starting point for the theory of homological projective duality. Therefore it is natural to ask what will be the homologically projectively dual for $\text{Gr}(2, n)$, $\text{SGr}(2, 2m)$, and $O\text{Gr}(2, 2m + 1)$ with respect to the Lefschetz exceptional collections considered here. In [8] we answer this question for the Grassmannians $\text{Gr}(2, 6)$ and $\text{Gr}(2, 7)$ and formulate a conjectural answer for $\text{Gr}(2, n)$ for all $n$. Since the symplectic Grassmannian $S\text{Gr}(2, 2m)$ is a hyperplane section of $\text{Gr}(2, 2m)$, Ref. [8] also gives an answer for $S\text{Gr}(2, 2m)$. For the orthogonal Grassmannian $O\text{Gr}(2, 2m + 1)$ the question is still open.

The paper is organized as follows. In Section 2 we recall the definition and some properties of Lefschetz exceptional collections, and in Section 3 we recall the Borel–Bott–Weil theorem. In Section 4 we show that (1) and (3) are full exceptional collections on Grassmannians $\text{Gr}(2, 2m)$ and $\text{Gr}(2, 2m + 1)$, respectively, and give a proof of the fullness of these collections by induction.
on $m$, demonstrating a method used later in the case of isotropic Grassmannians. In Section 5 we show that (2) is a full exceptional collection on the symplectic isotropic Grassmannians $\text{SGr}(2,2m)$. In Section 6 we develop a theory of spinor bundles on homogeneous spaces of orthogonal groups. Finally, in Section 7 we show that (4) is a full exceptional collection on the orthogonal isotropic Grassmannians $\text{OGr}(2,2m+1)$.

2. Lefschetz exceptional collections

Let $X$ be a smooth projective algebraic variety over a field $k$ with an ample line bundle $\mathcal{O}_X(1)$. We denote by $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on $X$.

Definition 2.1 (cf. [9]). A Lefschetz collection in $\mathcal{D}^b(X)$ with respect to the line bundle $\mathcal{O}_X(1)$ is a collection of objects of $\mathcal{D}^b(X)$ which has a block structure

$$\begin{pmatrix}
E_1, E_2, \ldots, E_{\lambda_0}, E_1(1), E_2(1), \ldots, E_{\lambda_1}(1), \ldots, E_1(i-1), E_2(i-1), \ldots, E_{\lambda_{i-1}}(i-1)
\end{pmatrix},$$

where $\lambda = (\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{i-1} > 0)$ is a non-increasing sequence of positive integers (the support partition of the Lefschetz collection).

In other words, a collection is Lefschetz with support partition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{i-1})$ if it splits into $i$ blocks of length $\lambda_0$, $\lambda_1$, $\ldots$, $\lambda_{i-1}$ such that the $k$th block consists of the first $\lambda_{k-1}$ objects of the first block twisted by $\mathcal{O}_X(k-1)$. A Lefschetz decomposition is uniquely determined by its first block $(E_1, E_2, \ldots, E_{\lambda_0})$ and its support partition $\lambda$.

Recall that a collection $(E_1, E_2, \ldots, E_n)$ of objects in $\mathcal{D}^b(X)$ is called exceptional if

1. for each $i$ the object $E_i$ is exceptional, that is, $\text{Hom}(E_i, E_i) = k$, $\text{Ext}^p(E_i, E_i) = 0$ for $p \neq 0$;
2. for all $i < j$ we have $\text{Ext}^*(E_j, E_i) = 0$.

The symmetry of Lefschetz collections simplifies the verification of exceptionality.

Lemma 2.2. A Lefschetz collection $E_\bullet$ with support partition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{i-1})$ is exceptional if and only if

1. its first block $(E_1, E_2, \ldots, E_{\lambda_0})$ is an exceptional collection, and
2. $\text{Ext}^*(E_p, E_q(-k)) = 0$ for $1 \leq k \leq i-1$, $1 \leq p \leq \lambda_k$, and $1 \leq q \leq \lambda_0$.

Proof. It suffices to note that $\text{Ext}^*(E_p(k), E_q(l)) = \text{Ext}^*(E_p, E_q(l-k))$. 

Example 2.3. Now let us give several examples of Lefschetz exceptional collections

- Any exceptional collection is a 1-block Lefschetz collection.
- For any $d > 0$ the standard exceptional collection $(\mathcal{O}_{\mathbb{P}n}, \mathcal{O}_{\mathbb{P}n}(1), \ldots, \mathcal{O}_{\mathbb{P}n}(n))$ on $\mathbb{P}n$ is Lefschetz with respect to $\mathcal{O}_{\mathbb{P}n}(d)$ with support partition $\lambda = (d, d, \ldots, d, r)$, where $n+1 = gd+r$, $0 < r \leq d$.
- Kapranov’s exceptional collection on an odd-dimensional quadric $$(\mathcal{O}_{Q^n}, S, \mathcal{O}_{Q^n}(1), \ldots, \mathcal{O}_{Q^n}(n-1))$$
is Lefschetz with respect to \( O_Q^n(1) \) with support partition

\[
\lambda = (2, 1, \ldots, 1).
\]

Here \( S \) is the spinor bundle.

- Kapranov’s exceptional collection on an even-dimensional quadric

\[
(\mathcal{O}_Q^n, S^+, S^-, \mathcal{O}_Q^n(1), \mathcal{O}_Q^n(2), \ldots, \mathcal{O}_Q^n(n - 1))
\]

is Lefschetz with respect to \( \lambda = (3, 1, \ldots, 1) \).

Here \( S^+ \) and \( S^- \) are the spinor bundles on \( Q^n \). However, this Lefschetz collection is not minimal. Indeed, it is easy to see that the mutation of \( S^- \) through \( \mathcal{O}_Q^n(1) \) is isomorphic to \( S^+(1) \), and hence the collection on \( Q^n \),

\[
(\mathcal{O}_Q^n, S^+, \mathcal{O}_Q^n(1), S^+(1), \mathcal{O}_Q^n(2), \ldots, \mathcal{O}_Q^n(n - 1)).
\]

is also a full exceptional Lefschetz collection with respect to \( \mathcal{O}_Q^n(1) \) but with another support partition

\[
\lambda' = (2, 2, 1, \ldots, 1).
\]

A simple but very useful property is that a Lefschetz exceptional collection in \( D^b(X) \) with respect to \( \mathcal{O}_X(1) \) gives a Lefschetz exceptional collection on any hyperplane section of \( X \). Explicitly, if we remove the first block of a Lefschetz exceptional collection and then restrict the rest to a hyperplane, then we obtain a Lefschetz exceptional collection.

**Proposition 2.4** [9]. Let

\[
(E_1, \ldots, E_{\lambda_0}, E_1(1), \ldots, E_{\lambda_1}(1), \ldots, E_1(i - 1), \ldots, E_{\lambda_{n - 1}}(i - 1))
\]

be a Lefschetz exceptional collection in \( D^b(X) \) with support partition \( \lambda \). Let \( Y \subset X \) be a hyperplane section with respect to \( \mathcal{O}_X(1) \). Then

\[
(E_1, \ldots, E_{\lambda_1}, E_1(1), \ldots, E_{\lambda_2}(1), \ldots, E_1(i - 2), \ldots, E_{\lambda_{n - 1}}(i - 2))
\]

is a Lefschetz exceptional collection in \( D^b(Y) \) with support partition \( \lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_{i - 1}) \).

**Proof.** We have \( \operatorname{Ext}^*_Y(E_p, E_q(-k)) = H^*(Y, E^*_p \otimes E_q(-k)) \). Since \( Y \) is a hyperplane section of \( X \) with respect to \( \mathcal{O}_X(1) \), we have a resolution

\[
0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.
\]

Tensoring it with \( E^*_p \otimes E_q(-k) \) we obtain a long exact sequence

\[
\cdots \to H^*(X, E^*_p \otimes E_q(-k)) \to H^*(Y, E^*_p \otimes E_q(-k)) \to H^{*+1}(X, E^*_p \otimes E_q(-k - 1)) \to \cdots.
\]

Taking \( k = 0 \) we deduce that \( \operatorname{Ext}^*_Y(E_p, E_q) = \operatorname{Ext}^*_X(E_p, E_q) \) for \( 1 \leq p, q \leq \lambda_1 \); hence the collection \( (E_1, E_2, \ldots, E_{\lambda_1}) \) on \( Y \) is exceptional. Taking \( 1 \leq k \leq i - 2 \) we deduce that \( \operatorname{Ext}^*_Y(E_p, E_q(-k)) = 0 \) for \( 1 \leq p \leq \lambda_{k + 1} \) and \( 1 \leq q \leq \lambda_1 \). By Lemma 2.2 this means that the desired Lefschetz collection on \( Y \) is exceptional. \( \square \)
3. The Borel–Bott–Weil Theorem

The Borel–Bott–Weil theorem computes the cohomology of line bundles on the flag variety of a semisimple algebraic group. It can also be used to compute the cohomology of equivariant vector bundles on Grassmannians. We restrict here to the case of the group \( GL(V) \) (see, however, Remark 3.2).

Let \( V \) be a vector space of dimension \( n \). The standard identification of the weight lattice of the group \( GL(V) \) with \( \mathbb{Z}^n \) takes the \( k \)th fundamental weight \( \pi_k \) (the highest weight of the representation \( \Lambda^k V \)) to the vector \((1, 1, \ldots, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^n \) (the first \( k \) entries are 1, and the last \( n - k \) are 0). Under this identification, the cone of dominant weights of \( GL(V) \) gets identified with the set of non-increasing sequences \( \alpha = (a_1, a_2, \ldots, a_n) \) of integers. For such \( \alpha \) we denote by \( \Sigma^\alpha V = \Sigma^{a_1, a_2, \ldots, a_n} V \) the corresponding representation of \( GL(V) \). Note that \( \Sigma^{1,1,\ldots,1} V = \det V \).

Similarly, given a vector bundle \( E \) of rank \( n \) on a scheme \( S \), we consider the corresponding principal \( GL(n) \)-bundle on \( S \) and denote by \( \Sigma^\alpha E \) the vector bundle associated with the \( GL(n) \)-representation of highest weight \( \alpha \).

The group \( S_n \) of permutations acts naturally on the weight lattice \( \mathbb{Z}^n \). Denote by \( \ell : S_n \to \mathbb{Z} \) the standard length function. Note that for every \( \alpha \in \mathbb{Z}^n \) there exists a permutation \( \sigma \in S_n \) such that \( \sigma(\alpha) \) is non-increasing. If all entries of \( \alpha \) are distinct then such \( \sigma \) is unique and \( \sigma(\alpha) \) is strictly decreasing.

Let \( X \) be the flag variety of \( GL(V) \). Let \( L_\alpha \) denote the line bundle on \( X \) corresponding to the weight \( \alpha \) (such that \( L_{\pi_k} \) is the pullback of \( \mathcal{O}_{\mathbb{P}(\Lambda^k V)}(1) \) under the natural projection \( X \to \mathbb{P}(\Lambda^k V) \)).

Denote by
\[
\rho = (n, n - 1, \ldots, 2, 1)
\]
half the sum of the positive roots of \( GL(V) \). The corresponding line bundle \( L_\rho \) is the square root of the anticanonical line bundle.

The Borel–Bott–Weil theorem computes the cohomology of line bundles \( L_\alpha \) on \( X \).

**Theorem 3.1** ([4]). Assume that all entries of \( \alpha + \rho \) are distinct. Let \( \sigma \) be the unique permutation such that \( \sigma(\alpha + \rho) \) is strictly decreasing. Then
\[
H^k(X, L_\alpha) = \begin{cases} \Sigma^{\sigma(\alpha + \rho) - \rho} V^* & \text{if } k = \ell(\sigma), \\ 0 & \text{otherwise.} \end{cases}
\]
If not all entries of \( \alpha + \rho \) are distinct then \( H^*(X, L_\alpha) = 0 \).

**Remark 3.2.** The Borel–Bott–Weil theorem is true for any semisimple algebraic group. One should replace in the statement \( \mathbb{Z}^n \) by the weight lattice, the set of strictly decreasing sequences by the interior of the dominant cone, \( \rho \) by half the sum of the positive roots, and the group \( S_n \) by the Weil group.

Now consider a Grassmannian \( Gr(k,V) \). Let \( U \subset V \otimes \mathcal{O}_{Gr(k,V)} \) denote the tautological sub-bundle of rank \( k \). Denote by \( V/\mathcal{U} \) the corresponding quotient bundle and by \( U^* \) its dual, so that we have the following (mutually dual) exact sequences
\[
0 \to U \to V \otimes \mathcal{O}_{Gr(k,V)} \to V/\mathcal{U} \to 0, \quad 0 \to U^* \to V^* \otimes \mathcal{O}_{Gr(k,V)} \to U^* \to 0.
\]
Note that \( \Sigma^{1,1,\ldots,1} U^* \cong \Sigma^{-1,-1,\ldots,-1} U^* \) is the positive generator of \( \text{Pic} Gr(k,V) \). Let \( \pi : X \to Gr(k,V) \) denote the canonical projection from the flag variety to the Grassmannian.
Assume that all entries of \( \alpha \) vector space then

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(= 1, and hence

\[ \text{to compute} \]

\[ \text{Ext} \]

\[ \text{The second condition implies that either} \]

\[ X \]

\[ k \]

\[ 2 \]

\[ \text{since} \]

\[ l \]

\[ k \]

\[ \text{and other cohomologies are zero. Combining all this we deduce the lemma.} \]

\[ \text{so it suffices to compute} \]

\[ H \]

\[ l \]

\[ \text{Proof. First of all, we have} \]

\[ \text{Ext}^p(S^{l_1}U^*, S^{l_2}U^*(-k)) = \begin{cases} S^{l_2-l_1}V^*, & \text{if} \ l_1 \leq l_2, k = 0, \text{and} \ p = 0, \\ k, & \text{if} \ l_1 = l_2 = n/2 - 1, k = n/2, \text{and} \ p = n - 2, \\ 0, & \text{otherwise.} \end{cases} \]

\[ \text{Further, by the Littlewood–Richardson rule we have} \]

\[ S^{l_1}U \otimes S^{l_2}U^*(-k) = \bigoplus_{t=0}^{\min\{l_1, l_2\}} S^{l_2-k-t, -l_1-k+t}U^* = S^{l_2-k, -l_1-k}U^* \oplus S^{l_1-1}U \otimes S^{l_2-1}U^*(-k), \]

\[ \text{so it suffices to compute} H^*(X, \Sigma^*U^*) \text{ for} \ \alpha = (l_2 - k, -l_1 - k, 0, 0, \ldots, 0). \text{Note that} \]

\[ \alpha + \rho = (n + l_2 - k, n - l_1 - k - 1, n - 2, n - 3, \ldots, 1). \]

Assume that all entries of \( \alpha + \rho \) are distinct. This is equivalent to

\[ n + l_2 - k \notin \{n - 2, n - 3, \ldots, 1\} \quad \text{and} \quad n - l_1 - k - 1 \notin \{n - 2, n - 3, \ldots, 1\}. \]

The second condition implies that either \( n - l_1 - k - 1 = n - 1, \) that is \( l_1 = k = 0, \) or \( n - l_1 - k - 1 \leq 0, \) that is \( l_1 + k \geq n - 1. \) In the first case, we get \( \alpha + \rho = (n + l_2, n - 1, n - 2, \ldots, 1), \) \( \sigma = 1, \) and hence \( H^0(X, \Sigma^*U^*) = S^{l_2}V^* \) and other cohomologies are zero. In the second case, since \( l_1 \leq n/2 - 1 \) it follows that \( k \geq n/2. \) Since \( l_2 \leq n/2 - 1 \) we have \( n + l_2 - k \leq n - 1, \) and since \( k \leq n - 1 \) we have \( n + l_2 - k \geq 1. \) Therefore, the first of the above conditions implies that \( n + l_2 - k = n - 1, \) and hence \( l_1 = l_2 = n/2 - 1 \) and \( k = n/2. \) In this case \( \alpha + \rho = (n - 1, 0, n - 2, n - 3, \ldots, 1), \ell(\sigma) = n - 2, \sigma(\alpha + \rho) - \rho = (-1, -1, \ldots, -1), \) and hence \( H^{n-2}(X, \Sigma^*U^*) = k \) and other cohomologies are zero. Combining all this we deduce the lemma.

\[ \square \]

4. **Usual Grassmannian**

Consider the Grassmannian \( X = \text{Gr}(2, W) \) of two-dimensional subspaces in an \( n \)-dimensional vector space \( W. \) Let \( \mathcal{U} \) denote the tautological rank 2 sub-bundle on \( X = \text{Gr}(2, W). \) We will
distinguish between the cases of even and odd $n$. Let

$$m = \left\lfloor \frac{n}{2} \right\rfloor$$

such that either $n = 2m$ or $n = 2m + 1$. If $n = 2m$ then we consider the collection (1) on $X$, and if $n = 2m + 1$ then we consider the collection (3) on $X$. These are Lefschetz collections with the first block

$$(O_X, U^*, S^2U^*, \ldots, S^{m-1}U^*)$$

and with the support partition

$$\lambda = \begin{cases} 
(m, m, \ldots, m) & \text{if } n = 2m + 1, \\
(m, m, \ldots, m, m-1, m-1, \ldots, m-1) & \text{if } n = 2m.
\end{cases}$$

In other words, the collections consist of vector bundles $S^lU^*(k)$ with integers $(k, l)$ from the set

$$\Upsilon_n = \{(k, l) \in \mathbb{Z}^2 \mid 0 \leq k \leq n - 1, 0 \leq l \leq m - 1 \text{ and } l \leq m - 2 \text{ for } k \geq m \text{ and even } n\}.$$ (7)

It is interesting to compare these collections with the standard Kapranov’s collections. The latter also consist of vector bundles $S^lU^*(k)$ but with other restrictions on possible values of $(k, l)$, namely $0 \leq k, l$ and $k + l \leq n - 2$. In Figure 1, the triangles corresponding to Kapranov’s exceptional collections are drawn together with the regions $\Upsilon_n$.

![Figure 1. Regions $\Upsilon_n$](image)

The main result of this section is the following theorem.

**Theorem 4.1.** Let $X = \text{Gr}(2, n)$, and let $\Upsilon_n$ be the set defined in (7). Then the Lefschetz collection $\{S^lU^*(k) \mid (k, l) \in \Upsilon_n\}$ is a full exceptional collection in $D^b(X)$.

Certainly, the collection $\{S^lU^*(k) \mid (k, l) \in \Upsilon_n\}$ can be obtained by a sequence of mutations from Kapranov’s exceptional collection (this is really not too complicated; one should consider complexes (11) constructed below). However, we prefer to use an inductive argument, since it can (and will) be applied for the symplectic and orthogonal Grassmannians as well.

For a start, we must check that the collection is exceptional. This is easily done by the Borel–Bott–Weil theorem.

**Lemma 4.2.** The Lefschetz collection $\{S^lU^*(k) \mid (k, l) \in \Upsilon_n\}$ is exceptional in $D^b(X)$.
Proof. Combine Lemmas 2.2 and 3.5.

It remains to prove the fullness of the collection, and we begin with the following.

Besides the set $\Upsilon_n$, consider also the set

$$\tilde{\Upsilon}_{n-1} = \{(k, l) \in \mathbb{Z}^2 \mid 0 \leq k \leq n-1, 0 \leq l \leq m-1, \text{ and } l \leq m-2 \text{ for } k \geq m-1 \text{ and odd } n\}. \quad (8)$$

In Figure 2, the region $\Upsilon_n$ is drawn together with the region $\tilde{\Upsilon}_{n-1}$.

Figure 2. Regions $\tilde{\Upsilon}_n$

Lemma 4.3. For any $(k, l) \in \tilde{\Upsilon}_{n-1}$ the vector bundle $S^lU^*(k)$ lies in the triangulated category generated by the Lefschetz collection $\{S^lU^*(k) \mid (k, l) \in \Upsilon_n\}$ in $D^b(\text{Gr}(2, W))$.

Proof. Let $n = 2m$. Then $\tilde{\Upsilon}_{n-1} \setminus \Upsilon_n = \{(m, m-1), (m+1, m-1), \ldots, (2m-1, m-1)\}$. Therefore, we have to check that the vector bundles $S^{m-1}U^*(m), S^{m-1}U^*(m+1), \ldots, S^{m-1}U^*(2m-1)$ lie in the triangulated category generated by the Lefschetz collection $\{S^lU^*(k) \mid (k, l) \in \Upsilon_n\}$ on $\text{Gr}(2, W)$.

Consider the (dual) tautological exact sequence $0 \rightarrow U^k \rightarrow W^* \otimes \mathcal{O}_X \rightarrow U^* \rightarrow 0$ on $X = \text{Gr}(2, W)$. It induces the long exact sequence

$$0 \rightarrow \Lambda^kU^k \rightarrow \Lambda^kW^* \otimes \mathcal{O}_X \rightarrow \Lambda^{k-1}W^* \otimes U^* \rightarrow \cdots \rightarrow W^* \otimes S^{k-1}U^* \rightarrow S^{k-2}U^* \rightarrow 0 \quad (9)$$

for any $0 \leq k \leq n-2$. Dualizing, using isomorphisms $U^{k-2} \cong W/U, S^lU \cong S^lU^* \otimes \mathcal{O}_X(-l)$ and replacing $k$ by $n-2-k$ we obtain the exact sequence

$$0 \rightarrow S^{n-2-k}U^*(k+2-n) \rightarrow W \otimes S^{n-3-k}U^*(k+3-n) \rightarrow \cdots \rightarrow \Lambda^{n-3-k}W \otimes U^*(-1) \rightarrow \Lambda^{n-2-k}W \otimes \mathcal{O}_X \rightarrow \Lambda^{n-2-k}(W/U) \rightarrow 0. \quad (10)$$

On the other hand, we have an isomorphism

$$\Lambda^kU^k \cong \Lambda^{n-2-k}(W/U) \otimes \mathcal{O}_X(-1).$$

Using this isomorphism for gluing sequence (9) twisted by $n-k-1$ with sequence (10) twisted by $n-k-2$, we obtain the following exact sequence

$$0 \rightarrow S^{n-2-k}U^* \rightarrow W \otimes S^{n-3-k}U^*(1) \rightarrow \cdots \rightarrow \Lambda^{n-3-k}W \otimes U^*(n-k-3) \rightarrow \Lambda^{n-2-k}W \otimes \mathcal{O}_X(n-k-2) \rightarrow \Lambda^kW^* \otimes \mathcal{O}_X(n-k-1) \rightarrow \Lambda^{k-1}W^* \otimes U^*(n-k-1) \rightarrow \cdots \rightarrow W^* \otimes S^{k-1}U^*(n-k-1) \rightarrow S^kU^*(n-k-1) \rightarrow 0. \quad (11)$$

Take $k = m-1$. Then sequence (11) gives a decomposition for $S^{m-1}U^*(m)$ with respect to the Lefschetz collection. Twisting this sequence by $\mathcal{O}_X(1), \ldots, \mathcal{O}_X(m-1)$ we obtain decompositions also for $S^{m-1}U^*(m+1), \ldots, S^{m-1}U^*(2m-1)$. 
Lemma 3.1] there exists an object of $U_X$ is the Grassmannian resolution of the structure sheaf. Therefore we have the following resolution of the structure sheaf $O_{X_\phi}$ on $X$:

$$0 \to O_X(-1) \to U \to O_X \to i_{\phi*}O_{X_\phi} \to 0, \quad (12)$$

where $i_{\phi*}: X_\phi \to X$ is the embedding.

**Proof.** The first part is evident. For the second part, we note that any non-zero section $\phi$ of $U^*$ is regular since $\dim X_\phi = 2(n - 1) - 4 = \dim X - 2$, and so the sheaf $i_{\phi*}O_{X_\phi}$ admits a Koszul resolution which takes the form (12).

Now we are ready for the proof of the theorem. We use induction on $n$. The base of induction, $n = 3$, is clear. Indeed, in this case $X = \text{Gr}(2, W) = \mathbb{P}^2$, and the Lefschetz collection takes the form $(O_X, O_X(1), O_X(2))$, which is well known to be full.

Now assume that the fullness of the corresponding Lefschetz collection is already proved for $n - 1$. Assume also that the Lefschetz collection for $n$ is not full. Then by [2, Theorem 3.2(a) and Lemma 3.1] there exists an object $F \in \mathcal{D}^b(X)$ right orthogonal to all bundles in the collection, and hence, by Lemma 4.3, to all $S^lU^*(k)$ with $(k, l) \in \tilde{Y}_{n-1}$; that is,

$$0 = \text{RHom}(S^lU^*(k), F) = H^*(X, S^lU(-k) \otimes F) \quad \text{for all } (k, l) \in \tilde{Y}_{n-1}.$$

Let us check that $i^*_{\phi*}F = 0$ for any $0 \neq \phi \in W^*$. For this we take any $(k, l) \in \tilde{Y}_{n-1}$ and tensor the resolution (12) by $S^lU(-k) \otimes F$. Taking into account the isomorphism

$$(S^lU(-k) \otimes F) \otimes i_{\phi*}O_{X_\phi} \cong i_{\phi*}i^*_{\phi}(S^lU(-k) \otimes F) \cong i_{\phi*}(S^lU(-k) \otimes i^*_{\phi}(F)),$$

we get a resolution

$$0 \to S^lU(-k - 1) \otimes F \to S^lU(-k) \otimes U \otimes F \to S^lU(-k) \otimes F \to i_{\phi*}(S^lU(-k) \otimes i^*_{\phi}(F)) \to 0.$$

Now note that for $(k, l) \in \tilde{Y}_{n-1}$ we have

$$(k + 1, l), (k, l + 1), (k, l) \in \tilde{Y}_{n-1}, \quad \text{and also} \quad (k + 1, l - 1) \in \tilde{Y}_{n-1} \quad \text{if } l \geq 1.$$

Since $S^lU(-k) \otimes U = S^{l+1}U(-k) \oplus S^{l-1}U(-k)$ (the second summand vanishes if $l = 0$), it follows that the cohomology on $X$ of the first three terms of the above complex vanishes. Therefore we have

$$\text{RHom}_{X_\phi}(S^lU^*(k), i^*_{\phi*}F) = H^*(X_\phi, S^lU(-k) \otimes i^*_{\phi}(F)) = 0 \quad \text{for all } (k, l) \in \tilde{Y}_{n-1}.$$

Thus $i^*_{\phi*}F$ lies in the right orthogonal to the subcategory of $\mathcal{D}^b(X_\phi)$ generated by the exceptional collection $\{S^lU^*(k) \mid (k, l) \in \tilde{Y}_{n-1}\}$ which, by the induction hypothesis, is full. Hence indeed $i^*_{\phi*}F = 0$. Therefore we conclude by the following lemma.

**Lemma 4.5.** If for $F \in \mathcal{D}^b(X)$ we have $i^*_{\phi*}F = 0$ for any $0 \neq \phi \in W^*$, then $F = 0$. 

Proof. Assume that $F \neq 0$. Let $q$ be the maximal integer such that $\mathcal{H}^q(F) \neq 0$, take a point $x \in \text{supp} \mathcal{H}^q(F)$, and choose $0 \neq \phi \in W^*$ such that $x \in X_\phi$ (this is equivalent to the vanishing of a linear function $\phi$ on the two-dimensional subspace of $W$ corresponding to $x \in X = \text{Gr}(2, W)$). Since the functor $i^*_\phi$ is left-exact it easily follows that $\mathcal{H}^q(i^*_\phi F) \neq 0$, and so $i^*_\phi F \neq 0$.

Thus we have proved that the desired collection is indeed full.

5. Symplectic Grassmannian

Consider the isotropic Grassmannian $X = \text{SGr}(2, W)$ of two-dimensional subspaces in a symplectic vector space $W$ of dimension $2m$. Note that $\text{SGr}(2, W)$ is a hyperplane section of the usual Grassmannian $\text{Gr}(2, W)$; the hyperplane corresponds to the symplectic form $\omega \in \Lambda^2 W^* = H^0(\text{Gr}(2, W), \mathcal{O}(1))$. Let $\mathcal{U}$ denote the restriction of the tautological rank 2 subbundle from $\text{Gr}(2, W)$ to $X = \text{SGr}(2, W)$. Restricting the Lefschetz exceptional collection on $\text{Gr}(2, W)$ constructed in the previous section and using Proposition 2.4, we obtain the Lefschetz collection on $\text{SGr}(2, W)$. This is a Lefschetz collection with the first block (5) and the support partition

$$\lambda^S = (m, m, \ldots, m, m-1, m-1, \ldots, m-1). \quad (13)$$

In other words, the collection consists of vector bundles $S^l \mathcal{U}^*(k)$ with integers $(k, l)$ from the set

$$\Upsilon^S_{2m} = \{(k, l) \in \mathbb{Z}^2 \mid 0 \leq k \leq 2m - 2, 0 \leq l \leq m - 1, \text{ and } l \leq m - 2 \text{ for } k \geq m - 1\}. \quad (14)$$

In Figure 3, we draw the regions $\Upsilon_{2m}$ and $\Upsilon^S_{2m}$ in the $(k, l)$-plane.

Consider the set

$$\tilde{\Upsilon}^S_{2m-2} = \{(k, l) \in \mathbb{Z}^2 \mid 0 \leq k \leq 2m - 2, 0 \leq l \leq m, \text{ and } l \leq m - 1 \text{ for } k \geq m - 2\}. \quad (15)$$

In Figure 4, the region $\Upsilon^S_{2m}$ is drawn together with the region $\tilde{\Upsilon}^S_{2m-2}$.

The small circle corresponds to the bundle $S^{m-1} \mathcal{U}^*(m-1)$ which plays a special role as will be seen later.
Lemma 5.2. For any $(k, l) \in \tilde{\mathcal{Y}}^S_{2m-2}$ the vector bundle $S^lU^*(k)$ lies in the triangulated category generated by the Lefschetz collection $\{S^lU^*(k) \mid (k, l) \in \tilde{\mathcal{Y}}^S_{2m}\}$ in $D^b(\text{SGr}(2, W))$.

Proof. Note that

$$\tilde{\mathcal{Y}}^S_{2m-2} \setminus \mathcal{Y}_{2m} = \{(0, m), (1, m), \ldots, (m - 3, m)\} \cup \{(m - 1, m - 1)\}$$

$$\cup \{(m - 1, 1), \ldots, (2m - 2, m - 1)\}.$$

Therefore, we have to check that the vector bundles

$$S^mU^*, S^mU^*(1), \ldots, S^mU^*(m - 3), S^{m-1}U^*(m - 1), S^{m-1}U^*(m), \ldots, S^{m-1}U^*(2m - 2)$$

lie in the triangulated category generated by the Lefschetz collection $\{S^lU^*(k) \mid (k, l) \in \tilde{\mathcal{Y}}^S_{2m}\}$. Consider the restriction to $\text{SGr}(2, W) \subset \text{Gr}(2, W)$ of the exact sequence (11) with $k = m - 2$. It gives a decomposition of $S^mU^*$ with respect to the Lefschetz collection. Twisting this sequence by $O_X(1), \ldots, O_X(m - 3)$ we obtain decompositions also for $S^mU^*(1), \ldots, S^mU^*(m - 3)$. Similarly, taking $k = m - 1$ we obtain a decomposition of $S^{m-1}U^*(m)$ with respect to the Lefschetz collection. Twisting this sequence by $O_X(1), \ldots, O_X(m - 2)$ we obtain decompositions also for $S^{m-1}U^*(m + 1), \ldots, S^{m-1}U^*(2m - 2)$. It remains only to find a decomposition for the vector bundle $S^{m-1}U^*(m - 1)$. This is done in the following proposition. 

\begin{proposition}
On $\text{SGr}(2, W)$ there exists a bicomplex

$$\begin{array}{ccccccc}
S^{m-1}U^* & \longrightarrow & S^mU^* & \longrightarrow & S^{m-1}U^* & \longrightarrow & S^mU^*\\
\vee & & \vee & & \vee & & \vee \\
W^* \otimes S^{m-2}U^*(1) & \longrightarrow & S^{m-1}U^*(1) & \longrightarrow & S^mU^*(1) & \longrightarrow & S^{m-1}U^*(2) \\
\Lambda^2W^* \otimes S^{m-3}U^*(2) & \longrightarrow & W^* \otimes S^{m-2}U^*(2) & \longrightarrow & S^{m-1}U^*(2) & \longrightarrow & S^mU^*(2) \\
\Lambda^mW^* \otimes \mathcal{O}_{\text{Gr}}(m - 2) & \longrightarrow & \Lambda^{m-1}W^* \otimes S^2U^*(m - 2) & \longrightarrow & \Lambda^{m-2}W^* \otimes S^3U^*(m - 2) & \longrightarrow & \Lambda^{m-3}W^* \otimes S^4U^*(m - 2) & \longrightarrow & \Lambda^{m-4}W^* \otimes S^5U^*(m - 2) & \longrightarrow & \Lambda^{m-5}W^* \otimes S^6U^*(m - 2) \longrightarrow & S^{m-1}U^*(m - 2) \\
\Lambda^{m-1}W^* \otimes O_X(m - 1) & \longrightarrow & \Lambda^{m-2}W^* \otimes \mathcal{O}_{\text{Gr}}(m - 1) & \longrightarrow & \Lambda^{m-3}W^* \otimes S^2U^*(m - 1) & \longrightarrow & \Lambda^{m-4}W^* \otimes S^3U^*(m - 1) & \longrightarrow & \Lambda^{m-5}W^* \otimes S^4U^*(m - 1) & \longrightarrow & \Lambda^{m-6}W^* \otimes S^5U^*(m - 1) & \longrightarrow & S^{m-1}U^*(m - 1)
\end{array}$$

the total complex of which is exact.

Proof. First, consider the above diagram on the ambient Grassmannian $\text{Gr}(2, W)$ with rows being the twisted truncations of the exact sequences (9) and columns being the twisted truncations of the exact sequences (10), where we identify $\Lambda^kW$ with $\Lambda^kW^*$ via the symplectic form $\omega$. Certainly, this diagram is not commutative, but let us check that it commutes.
modulo $\omega$. More precisely, we will check that the compositions of arrows in the square

$$
\begin{array}{ccc}
\Lambda^k W^* \otimes S^{m-1-k}U^* & \longrightarrow & \Lambda^{k-1} W^* \otimes S^{m-k}U^* \\
\downarrow & & \downarrow \\
\Lambda^{k+1} W^* \otimes S^{m-2-k}U^*(1) & \longrightarrow & \Lambda^k W^* \otimes S^{m-1-k}U^*(1)
\end{array}
$$

which is a typical (up to a twist) square of the diagram, coincide modulo $\omega$. First of all, we apply the functor $\text{Hom}(S^{m-1-k}U^*, -)$ to this square and check that the resulting square of $\text{Hom}$ commutes modulo $\omega$. Indeed, since by the Borel–Bott–Weil theorem we have

\begin{align*}
\text{Hom}(S^{m-1-k}U^*, S^{m-1-k}U^*) &= k, \\
\text{Hom}(S^{m-1-k}U^*, S^{m-2-k}U^*(1)) &= W^*, \\
\text{Hom}(S^{m-1-k}U^*, S^{m-1-k}U^*(1)) &= W^* \otimes W^*,
\end{align*}

the square of $\text{Hom}$ takes the following form.

$$
\begin{array}{ccc}
\Lambda^k W^* & \longrightarrow & \Lambda^{k-1} W^* \otimes W^* \\
\downarrow_{\omega c \omega^{-1}} & & \downarrow_{\omega c \omega^{-1}} \\
\Lambda^{k+1} W^* \otimes W^* & \longrightarrow & \Lambda^k W^* \otimes W^* \otimes W^*
\end{array}
$$

Here $c$ is the canonical map, and $\omega c \omega^{-1}$ is the canonical map conjugated by $\omega$. Let $\{e_i\}$ be a base of $W$, and $\{f_i\}$ be the dual base of $W^*$. Then the compositions of arrows in this square act as follows.

\begin{align*}
\alpha & \mapsto c \sum (\alpha \vdash e_i) \otimes f_i \mapsto \sum (\alpha \vdash e_i) \otimes f_j \otimes \omega(e_j) \otimes f_i, \\
\alpha & \mapsto \omega c \omega^{-1} \sum (\alpha \otimes f_j) \otimes \omega(e_j) \mapsto \sum (\alpha \vdash e_i) \otimes f_j \otimes \omega(e_j) \otimes f_i \mapsto +(-1)^k \alpha \otimes (f_j \vdash e_i) \otimes f_i \otimes \omega(e_j),
\end{align*}

Here $\alpha \in \Lambda^k W^*$ and $\vdash$ denotes the convolution of a form and a vector. It remains to note that we have $f_j \vdash e_i = \delta_{ij}$ and $\sum f_i \otimes \omega(e_i) = \omega$, and hence indeed the difference of the compositions is given by the map $\Lambda^k W^* \rightarrow \Lambda^k W^* \otimes \Lambda^2 W^* \otimes W^* \rightarrow \Lambda^k W^* \otimes W^* \otimes W^* \rightarrow \Lambda^{k} W^*$. Finally, we note that the canonical maps $\text{Hom}(S^{m-1-k}U^*, F) \otimes S^{m-1-k}U^* \rightarrow F$ for $F = S^{m-1-k}U^*, S^{m-k}U^*, S^{m-2-k}U^*(1)$, and $S^{m-1-k}U^*(1)$ are surjective, and hence it follows that the squares of the constructed diagram commute modulo $\omega$.

Now we restrict the diagram to the isotropic Grassmannian $\text{SGr}(2, W)$. It follows that the squares now commute (because $\omega$ vanishes); so we got a bicomplex. It remains to check that its total complex is exact.

Consider the spectral sequences of this bicomplex. The first term of the first spectral sequence (the cohomology of rows) is concentrated at the left column by (9), which means that the total complex can have non-trivial cohomology only in the first $m$ terms. On the other hand, the first term of the second spectral sequence (the cohomology of columns) is concentrated at the bottom row by (10), which means that the total complex can have non-trivial cohomology only in the last $m$ terms. Combining these two observations we deduce that the total complex can have non-trivial cohomology only in the middle ($m$th) term. Finally, considering again the second spectral sequence and using (10) we see that this cohomology is a subsheaf of the sheaf $\Lambda^{m-1}(W/U) \otimes \mathcal{O}_X(m - 1)$, and hence is torsion free. However, computing the Euler characteristics of the total complex we see that its rank is zero, and hence the cohomology vanishes.

Another preparatory result is the following lemma.
Lemma 5.4. For any two-dimensional subspace \( \langle w_1, w_2 \rangle \subset W \cong W^* = H^0(\text{SGr}(2,W),\mathcal{U}^*) \) such that \( \omega(w_1, w_2) \neq 0 \), the zero locus of the corresponding section \( \phi = \phi_{w_1,w_2} \in H^0(\text{SGr}(2,W),\mathcal{U}^* \oplus \mathcal{U}^*) \) on \( X = \text{SGr}(2,W) \) is the isotropic Grassmannian \( X_{w_1,w_2} = \text{SGr}(2,\langle w_1, w_2 \rangle^\perp) \subset \text{SGr}(2,W) = X \). Moreover, we have the following resolution of the structure sheaf \( \mathcal{O}_{X_{w_1,w_2}} \) on \( X \):

\[
0 \rightarrow \mathcal{O}_X(-2) \rightarrow \mathcal{U}(-1) \oplus \mathcal{U}(-1) \rightarrow \mathcal{O}_X(-1)^{\oplus 3} \oplus S^2 \mathcal{U} \rightarrow \mathcal{U} \oplus \mathcal{U} \rightarrow \mathcal{O}_X \rightarrow 0, \tag{16}
\]

where \( \phi : X_{w_1,w_2} \rightarrow X \) is the embedding.

Proof. The first part is evident (one should only note that the restriction of \( \omega \) to the subspace \( \langle w_1, w_2 \rangle^\perp \) is non-degenerate provided that \( \omega(w_1, w_2) \neq 0 \)). For the second part, we note that any such section \( \phi = \phi_{w_1,w_2} \) of \( \mathcal{U}^* \oplus \mathcal{U}^* \) is regular since \( \dim X_{w_1,w_2} = 2(n - 2) - 5 = \dim X - 4 \), and so the sheaf \( \phi_* \mathcal{O}_{X_{w_1,w_2}} \) admits a Koszul resolution which takes the form (16).

Now we are ready for the proof of the theorem. We use induction on \( m \). The base of induction, \( m = 2 \), is clear. Indeed, in this case \( X = \text{SGr}(2,W) = Q^3 \), a three-dimensional quadric, and the Lefschetz collection takes the form \((\mathcal{O}_X, \mathcal{U}^*, \mathcal{O}_X(1), \mathcal{O}_X(2))\) which is well known to be full (actually, this is precisely Kapranov’s exceptional collection for \( Q^3 \)).

Now assume that the fullness of the corresponding Lefschetz collection is already proved for \( m - 1 \). Assume also that the Lefschetz collection for \( m \) is not full. Then by [2, Theorem 3.2(a) and Lemma 3.1] there exists an object \( F \in \mathcal{D}^b(X) \), right orthogonal to all bundles in the collection, and hence by Lemma 5.2

\[
0 = R\text{Hom}(S^l \mathcal{U}^*(k), F) = H^*(X, S^l \mathcal{U}(-k) \otimes F) \quad \text{for all} \ (k,l) \in \tilde{T}_2^{S_{2m-2}}.
\]

Let us check that \( i_{\phi}^* F = 0 \) for any \( \phi = \phi_{w_1,w_2} \) as in Lemma 5.4. For this we take any \( (k,l) \in \tilde{T}_2^{S_{2m-2}} \) and tensor the resolution (16) by \( S^l \mathcal{U}(-k) \otimes F \). Taking into account the isomorphism

\[
(S^l \mathcal{U}(-k) \otimes F) \otimes i_{\phi}^* \mathcal{O}_{X_{w_1,w_2}} \cong i_{\phi}^* (i_{\phi}^*(S^l \mathcal{U}(-k) \otimes F)) \cong i_{\phi}^* (S^l \mathcal{U}(-k) \otimes i_{\phi}^*(F))
\]

and noting that for \( (k,l) \in T_2^{S_{2m-2}} \) we have

\[
(k + 2, l), (k + 2, l - 1), (k + 1, l + 1), (k + 1, l),
(k + 2, l - 2), (k, l + 2), (k, l + 1), (k + 1, l - 1), (k, l) \in T_2^{S_{2m-2}},
\]

it follows that the cohomology on \( X \) of the first five terms of the above complex vanishes. Therefore we have

\[
R\text{Hom}_{X_{w_1,w_2}}(S^l \mathcal{U}^*(k), i_{\phi}^* F) = H^*(X_{w_1,w_2}, S^l \mathcal{U}(-k) \otimes i_{\phi}^*(F)) = 0 \quad \text{for all} \ (k,l) \in T_2^{S_{2m-2}}.
\]

Thus \( i_{\phi}^* F \) lies in the right orthogonal to the subcategory of \( \mathcal{D}^b(X_{w_1,w_2}) \) generated by the exceptional collection \( \{ S^l \mathcal{U}^*(k) \mid (k,l) \in T_2^{S_{2m-2}} \} \), which, by the induction hypothesis, is full. Hence indeed \( i_{\phi}^* F = 0 \). Therefore we conclude by the following lemma.

Lemma 5.5. If for \( F \in \mathcal{D}^b(X) \) we have \( i_{\phi}^* F = 0 \) for any two-dimensional subspace \( \langle w_1, w_2 \rangle \subset W \) such that \( \omega(w_1, w_2) \neq 0 \) then \( F = 0 \).

Proof. Assume that \( F \neq 0 \). Let \( q \) be the maximal integer such that \( \mathcal{H}^q(F) \neq 0 \); take a point \( x \in \text{supp} \mathcal{H}^q(F) \) and choose \( \langle w_1, w_2 \rangle \subset W \) such that \( x \in X_{w_1,w_2} \) (this is equivalent to the orthogonality of \( w_1 \) and \( w_2 \) with the two-dimensional subspace of \( W \) corresponding to...
Thus we have proved that the desired collection is indeed full.

Remark 5.6. The same argument allows to show that the Lefschetz exceptional collection on a smooth hyperplane section of \( \text{Gr}(2, 2m + 1) \) obtained by Proposition 2.4 from the Lefschetz collection (3) is full.

6. Spinor bundles

Our further aim is to construct an exceptional collection on the orthogonal isotropic Grassmannian \( \text{OGr}(2, 2m + 1) \). However, in this case, as in the case of quadrics, the restrictions of the tautological vector bundles from \( \text{Gr}(2, 2m + 1) \) do not give a full exceptional collection and we need to consider some analogs of spinor bundles. In this section, we construct spinor bundles on isotropic Grassmannians of a quadratic form, generalizing the definition of spinor bundles on quadrics and investigate their properties.

We start with a reminder on Clifford algebras. Let \( R \) be a commutative algebra over a field of zero characteristic, \( E \) a free \( R \)-module of rank \( n \), and \( q \in S^2E^* \), a quadratic form. The Clifford algebra of \( q \) is defined as the quotient of the tensor algebra of \( E \) by the following two-sided ideal (see [3])

\[
B_q = T^*(E)/\langle e_1 \otimes e_2 + e_2 \otimes e_1 - 2q(e_1, e_2)1 \rangle.
\]

The Clifford algebra is naturally \( \mathbb{Z}/2\mathbb{Z} \)-graded, \( B_q = B_q^0 \oplus B_q^1 \), the grading is induced by the \( \mathbb{Z} \)-grading of the tensor algebra. In what follows we will be mostly interested in \( B_q^0 \), the even part of the Clifford algebra. Sometimes, instead of the quadratic form we will use the underlying vector space as an index and sometimes (when the quadratic form and the space are clear) the index will be omitted. Note that as an \( R \)-module the Clifford algebra takes the form

\[
B_q = \Lambda^* E = R \oplus E \oplus \Lambda^2 E \oplus \ldots, \quad B_q^0 = \Lambda^+ E = R \oplus \Lambda^2 E \oplus \Lambda^4 E \oplus \ldots.
\]

Assume that \( n = 2m \) is even and the determinant of the quadratic form is invertible. Then it is well known (see [3]) that if \( q \) is neutral (that is, the space \( E \) of \( q \) can be decomposed into a sum of two isotropic \( R \)-submodules) then \( B^0 \) is isomorphic to a product of two matrix algebras. Indeed, let \( E = E_1 \oplus E_2 \) be a decomposition of \( E \) into a direct sum of two isotropic submodules. Then the form \( q \) gives an isomorphism \( E_2 \cong E_1^* \). Let \( S_+ \) and \( S_- \) denote the even and the odd parts of the exterior algebra of \( E_1 \), respectively:

\[
S_+ = \Lambda^+ E_1, \quad S_- = \Lambda^- E_1.
\]

The algebra \( B \) acts on \( S_+ \oplus S_- \) (the action of elements of \( E_1 \) is given by the wedge-product, while the action of \( E_2 \cong E_1^* \) is given by the convolution), and the summands are invariant with respect to the action of \( B^0 \). Thus we obtain a morphism of algebras \( B^0 \to \text{End}(S_+) \times \text{End}(S_-) \), which is actually an isomorphism. The simple \( B^0 \)-modules \( S_+ \) and \( S_- \) are called the half-spinor modules.

The spinor module \( S = S_+ \oplus S_- \) over \( B \) is self-dual (non-degenerate \( B \)-invariant pairing on \( S \cong \Lambda^*(E_1) \) is given by the determinant of the wedge-product). The duality isomorphism \( S_+ \oplus S_- \to S_+^* \oplus S_-^* \) gives a duality for half-spinor modules (interchanging them if \( m = n/2 \) is odd).

Similarly, assume that \( n = 2m + 1 \) is odd and the quadratic form is non-degenerate. Let \( E = E_1 \oplus E_2 \oplus Re \) be a decomposition of \( E \) into a direct sum of two isotropic submodules and orthogonal to them one-dimensional module. The multiplication by \( e \) gives a map
$E_1 \oplus E_2 \to B^0_q$ giving rise to an isomorphism $B_{q'} \to B^0_{q}$, where $q'$ is a quadratic form on $E_1 \oplus E_2$ given by the formula $q'(x) = -q(x)q(e)$. In particular, the spinor $B_{q'}$-module $S = \Lambda^* E_1$ acquires a structure of a $B^0_q$-module and it is known that the corresponding homomorphism $B^0_q \to \text{End}(S)$ is an isomorphism.

In what follows we are going to consider the odd-dimensional and the even-dimensional cases simultaneously (as far as it is possible). The differences between these cases discussed above suggest the use of the following convention. Let $\epsilon$ be an index taking values $+, -, $ or empty, with the following meaning. If $\epsilon = +$ (if $\epsilon = -$), then it means that we are considering the even-dimensional case and the object with this index corresponds to the half-spinor module $\epsilon$. On the other hand, if $\epsilon$ is empty then it just means that we are considering the odd-dimensional case. Having fixed this convention, we can write down the above results as follows:

$$B^0 \cong \prod_{\epsilon} \text{End}(S_\epsilon), \quad S^*_\epsilon \cong S_{\pm\epsilon}.$$

Below we will also need some results concerning the relation of a Clifford algebra (and its spinor modules) for a quadratic form and some of its subforms. Explicitly, assume that $U \subset E$ is an isotropic subspace, $\dim_q U = k$. Let $U^\perp$ denote the orthogonal complement to $U$ in $E$, so that $U \subset U^\perp$. Note that the initial non-degenerate quadratic form on $E$ induces a quadratic form $q'$ on $U^\perp \subset E$ with kernel $U$, and a non-degenerate quadratic form $q''$ on $U^\perp/U$. The embedding $U^\perp \subset E$ and the projection $U^\perp \to U^\perp/U$ are compatible with quadratic forms, and hence induce morphisms of Clifford algebras.

Assume that we have given a decomposition $E = E_1 \oplus E_2$ if $n$ is even, and a decomposition $E = E_1 \oplus E_2 \oplus Re$ if $n$ is odd, where $E_1, E_2$ are isotropic $R$-submodules in $E$ and $e \in E$ is orthogonal both to $E_1$ and $E_2$. Assume also that $U \subset E_1$. Then we obtain also a decomposition for $U^\perp/U; \ U^\perp/U = E_1/U \oplus E_2 \cap U^\perp$ if $n$ is even, and $U^\perp/U = E_1/U \oplus E_2 \cap U^\perp \oplus Re$ if $n$ is odd. Let $S_{U,\epsilon}$ denote the (half)-spinor module of the algebra $B^0_{U^\perp/U}$. We can consider both $S_\epsilon$ and $S_{U,\epsilon}$ as $B^0_{U^\perp}$-modules.

**Lemma 6.1.** There is a canonical filtration $F^U_k$ on the (half)-spinor $B^0_E$-module $S_\epsilon$ by $B^0_{U^\perp}$-submodules $0 = F^U_{k+1}S_\epsilon \subset F^U_k S_\epsilon \subset \ldots \subset F^U_k S_\epsilon \subset F^U_0 S_\epsilon = S_\epsilon$ such that $F^U_k S_\epsilon/F^U_{k+1} S_\epsilon \cong S_{U,(-1)^\epsilon} \otimes \Lambda^k U$.

**Proof.** Actually, the Clifford algebra $B^0_{U^\perp/U}$ is isomorphic to the semisimple part of the algebra $B^0_{U^\perp}$ (the radical of $B^0_{U^\perp}$ equals $U \cdot B^0_{U^\perp} \subset B^0_{U^\perp}$), and the desired filtration is just the radical filtration of the $B^0_{U^\perp}$-module $S_\epsilon$, $F^U_k S_\epsilon = \Lambda^k U \cdot (S_{U,(-1)^\epsilon})$. The quotients of the radical filtration are modules over the semisimple part of the algebra, and hence they are isomorphic to direct sums of (half)-spinor modules. To see that they have the form written up in the lemma we note that the short exact sequence

$$0 \longrightarrow U \longrightarrow E_1 \longrightarrow E_1/U \longrightarrow 0$$

induces a filtration on $S_\epsilon = \Lambda^* E_1$ with quotients $\Lambda^{(-1)^\epsilon}(E_1/U) \otimes \Lambda^k U = S_{U,(-1)^\epsilon} \otimes \Lambda^k U$. This filtration coincides with the radical filtration discussed above. \qed
LEMMA 6.2. Any choice of splitting $E_1 = U \oplus E'_1$ induces an isomorphism of the Clifford algebra $\mathcal{B}^{0}_{U \perp}$ with $\prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$. The isomorphism depends on a choice of splitting in such a way that the filtration on $\prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$ induced by the filtration

$$0 = F_1(n-k)/2+1 \mathcal{B}^0_{U \perp} \subset F_1(n-k)/2 \mathcal{B}^0_{U \perp} \subset \cdots \subset F_1 \mathcal{B}^0_{U \perp} \subset F_0 \mathcal{B}^0_{U \perp} = \mathcal{B}^0_{U \perp} = \bigoplus_{s \geq 0} \Lambda^{2s} U^\perp,$$

$$F_1 \mathcal{B}^0_{U \perp} = \bigoplus_{s \geq 1} \Lambda^{2s} U^\perp$$

on $\mathcal{B}^0_{U \perp}$ does not depend on a choice of splitting.

Proof. A choice of splitting

$$E_1 = U \oplus E'_1$$

(17)

gives a decomposition $U^\perp = U \oplus E'$, where $E' = E'_1 \oplus E_2 \cap U^\perp$ if $n$ is even, and $E' = E'_1 \oplus E_2 \cap U^\perp \oplus \text{Re}$ if $n$ is odd. Note that the embeddings $U \subset U^\perp$ and $E' \subset U^\perp$ induce embeddings of the Clifford algebras $\Lambda^* U = \mathcal{B} U \subset \mathcal{B} E_1$ and $\mathcal{B} E_2 \subset \mathcal{B} E_1$. These subalgebras commute and we have an isomorphism $\mathcal{B} U \cong \Lambda^* U \otimes \mathcal{B} E_1$, and $\mathcal{B} U \cong (\Lambda^* U \otimes \mathcal{B} E_1)^0$.

On the other hand, the splitting (17) gives a splitting of the filtration of Lemma 6.1 and furthermore, an isomorphism $S_\epsilon \cong (\Lambda^* U \otimes (\oplus S_{U,\epsilon}))^\epsilon$. Thus we have

$$\prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon) \cong \prod_\epsilon \text{Hom}
$$

$$\left(S_{U,\epsilon}, \left(\Lambda^* U \otimes \left(\bigoplus_\epsilon S_{U,\epsilon}\right)^0\right)\right)$$

$$\cong \left(\Lambda^* U \otimes \prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_{U,\epsilon})\right)^0 = (\Lambda^* U \otimes \mathcal{B} E_1)^0,$$

and as a consequence we obtain an isomorphism $\mathcal{B}^0_{U \perp} \cong \prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$. On the other hand, we have a canonical direct sum decomposition $\mathcal{B}^0_{U \perp} \cong \bigoplus \Lambda^{2s} U^\perp$, which via the constructed isomorphism gives a direct sum decomposition of $\prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$.

However, the direct sum decomposition depends on a choice of splitting (17). Therefore we are going to observe that the image of the filtration $F_1 \mathcal{B}^0_{U \perp}$ on $\prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$ does not depend on a choice of splitting. Indeed, recall that the isomorphism $\mathcal{B}^0_{U \perp} \cong \prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$ which we use to transport the filtration is actually a composition of two isomorphisms

$$F^0_{U \perp} \longrightarrow (\Lambda^* U \otimes \mathcal{B} E_1^0)^0 \longrightarrow \prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$$

It is clear that the first of these isomorphisms takes the direct sum decomposition of $\mathcal{B}^0_{U \perp}$ to the direct sum decomposition $(\Lambda^* U \otimes \mathcal{B} E_1^0)^0 = (\Lambda^* U \otimes \Lambda^* E'_1)^0 = \bigoplus_s (\bigoplus_{k+l=2s} \Lambda^k U \otimes \Lambda^l E'_1)^0$, and so we are interested how the second isomorphism affects the latter direct sum decomposition. Note that $\prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$ has a canonical structure of a right $(\Lambda^* U \otimes \mathcal{B} E_1^0)^0$-module, and that the above isomorphism is an isomorphism of right $(\Lambda^* U \otimes \mathcal{B} E_1^0)^0$-modules (we have canonical isomorphism $E'_1 \cong U^\perp / U$ compatible with quadratic forms, showing that $(\Lambda^* U \otimes \mathcal{B} E_1^0)^0$ is a free right $(\Lambda^* U \otimes \mathcal{B} E_1^0)^0$-module of rank 1). It follows that a change of splitting (17) results in an automorphism of the $(\Lambda^* U \otimes \mathcal{B} E_1^0)^0$-module structure on $\prod_\epsilon \text{Hom}(S_{U,\epsilon}, S_\epsilon)$, that is, it is given by the left action of an invertible element of $(\Lambda^* U \otimes \mathcal{B} E_1^0)^0$. A change of splitting can be written as $e' \mapsto e' + \phi(e')$, where $\phi : E'_1 \to U$ is an $R$-linear map. It is easy to check that the corresponding invertible element of $(\Lambda^* U \otimes \mathcal{B} E_1^0)^0$ equals to

$$(\Lambda^* \phi)^0 \in \text{Hom}(\Lambda^* E'_1, \Lambda^* U)^0 = (\Lambda^0(E'_1)^\ast \otimes \Lambda^0 U)^0 \cong (\Lambda^0 E'_2 \otimes \Lambda^0 U)^0 \subset (\Lambda^0 U \otimes \mathcal{B} E_1^0)^0.$$
Now let $W$ be an orthogonal vector space (that is, a vector space equipped with a non-degenerate quadratic form). Let $n = \dim W$, $m = [n/2]$. Consider the isotropic Grassmannians $\text{OGr}(m, W)$ of $m$-dimensional (maximal) isotropic subspaces in $W$. It is well known that for odd $n$ the Grassmannian $\text{OGr}(m, W)$ is connected, its Picard group is $\mathbb{Z}$, its positive generator is very ample, and the space of its global sections is canonically isomorphic to the spinor module $S$ over the even part of the Clifford algebra $\mathcal{B}_W^0$ (and the composition of embeddings $\text{OGr}(m, W) \to \text{Gr}(m, W) \subset \mathbb{P}(\Lambda^m W)$ is given by the twice generator of the Picard group). Similarly, for even $n$ the Grassmannian $\text{OGr}(m, W)$ has two connected components, their Picard groups are $\mathbb{Z}$, positive generators are very ample, and the spaces of their global sections are canonically isomorphic to the half-spinor modules $S_\pm$ over the even part of the Clifford algebra $\mathcal{B}_W^0$. Let $\mathcal{F}_m$ denote (the connected component of) the isotropic Grassmannian $\text{OGr}(m, W)$ corresponding to the (half)-spinor module $S_\epsilon$, so that

\[
\text{OGr}(m, W) = \mathcal{F}_m \quad \text{if } n = 2m + 1 \quad \text{and} \quad H^0(\mathcal{F}_m, \mathcal{O}_{\mathcal{F}_m}(1)) = S_\epsilon,
\]

where $\mathcal{O}_{\mathcal{F}_m}(1)$ is the positive generator of the Picard group of $\mathcal{F}_m$.

Consider also the other isotropic Grassmannians in $W$, denote $\mathcal{F}_k = \text{OGr}(k, W)$. Further, for any subset $I \subset \{1, 2, \ldots, m\}$, denote by $\mathcal{F}_I$ the incidence subvariety in (the connected component of) the product $\prod_{i \in I} \text{Gr}(i, W)$ of the isotropic Grassmannians and for $J \subset I$ denote by $\pi_J$ the projection $\mathcal{F}_I \to \mathcal{F}_J$. In this notation $\mathcal{F}_1 = \text{OGr}(1, W) = Q$ is the quadric in $\mathbb{P}(W)$ and $\mathcal{F}_2 = \text{OGr}(2, W)$ is the isotropic Grassmannian of two-dimensional subspaces which will be considered in the next section. Also, let $\mathcal{O}_{\mathcal{F}_1}(1)$ denote the ample generator of the Picard group of $\mathcal{F}_1$ and put

\[
\mathcal{O}_I(d_1, d_2, \ldots, d_k) := \pi_1^* \mathcal{O}_{\mathcal{F}_1}(d_1) \otimes \pi_2^* \mathcal{O}_{\mathcal{F}_2}(d_2) \otimes \cdots \otimes \pi_k^* \mathcal{O}_{\mathcal{F}_k}(d_k).
\]

Also, let $U_k$ denote the tautological rank $k$ sub-bundle in the trivial bundle $W \otimes \mathcal{O}_{\mathcal{F}_k}$, the restriction of the tautological sub-bundle from $\text{Gr}(k, W)$ to $\mathcal{F}_k$. Note that if $n$ is odd then $\mathcal{F}_{1,2,\ldots,m}$ is the flag variety of the group $O(W)$ and all $\mathcal{F}_I$ are the partial flag varieties. Similarly, if $n$ is even then the flag variety of the group $O(W)$ is the fiber product $\mathcal{F}_{1,2,\ldots,m-2,m} \times \mathcal{F}_{1,2,\ldots,m-2}$ and all $\mathcal{F}_I$ with $m - 1 \not\in I$ are the partial flag varieties.

Now take any $I \subset \{1, 2, \ldots, m-1\}$, consider the diagram

\[
\begin{array}{ccc}
\mathcal{F}_I & \xrightarrow{\pi_I} & \mathcal{F}_m \\
\downarrow & & \downarrow \\
\mathcal{F}_I, & & \mathcal{F}_m
\end{array}
\]

and define the spinor bundles on $\mathcal{F}_I$ by the formula

\[
S_{I_\epsilon} = \pi_I^* \pi_m^* \mathcal{O}_{\mathcal{F}_m}(1).
\]

Note that taking $I = \{1\}$ we obtain the usual spinor bundles on the quadric $Q = \mathcal{F}_1$ (see \[\text{[6]}\]), and taking $I = \emptyset$ we obtain the (half)-spinor modules $S_\epsilon$ (considered as vector bundles on $\mathcal{F}_0 = \text{Spec } k$). Note also that

\[
S_{I_\epsilon} \cong \pi_k^* S_{k_\epsilon}, \quad \text{where } k = \max\{i \in I\}.
\]

Indeed, this easily follows from the base change since $\mathcal{F}_{I,m} = \mathcal{F}_I \times \mathcal{F}_k \mathcal{F}_{k,m}$.

It is important that the spinor bundles can be considered as a generalization of spinor modules in a relative situation. Indeed, consider the vector bundle $U_k^* /U_k$ over $\mathcal{F}_k$. It carries a natural quadratic form induced by the quadratic form on $W$. Let $\mathcal{B}_k^0$ denote the corresponding sheaf of even parts of Clifford algebras on $\mathcal{F}_k$. The fiber of $\mathcal{B}_k^0$ over a point of $\mathcal{F}_k$ corresponding to an isotropic subspace $U \subset W$ is the Clifford algebra $\mathcal{B}_U^0$ and the fiber of $S_{k_\epsilon}$ is the
(half)-spinor module $S_{k,e}$. Therefore we can use local properties of spinor modules proved in Lemmas 6.1 and 6.2 to deduce the global properties of spinor bundles.

**Proposition 6.3.** The pullback of the (half)-spinor module $S_e$ to $\mathcal{G}_k$ admits a length $k + 1$ filtration $0 = F_{k+1}S_e \subset F_kS_e \subset \cdots \subset F_1S_e \subset F_0S_e = S_e \otimes \mathcal{O}_{\mathcal{G}_k}$ such that

$$F_iS_e / F_{i+1}S_e \cong S_{k,(-1)^i} \otimes \Lambda^i U_k.$$  

In particular, there is a short complex on $X = \mathcal{G}_2 = \operatorname{OGr}(2,W)$ of the form

$$0 \rightarrow S_{2,e} \otimes \mathcal{O}_X(-1) \rightarrow S_e \otimes \mathcal{O}_X \rightarrow S_{2,e} \rightarrow 0$$  

with the only non-trivial cohomology in the middle, isomorphic to $S_{2,-e} \otimes U_2$.

**Proof.** This is a global version of the local filtration of Lemma 6.1. \hfill \qedsymbol

Similarly, one can prove a relative version of this proposition.

**Proposition 6.4.** The pullback $\pi^*_e S_{k,e}$ of the spinor bundle to $\mathcal{G}_{k,l}$ admits a length $(l - k + 1)$ filtration $0 = F_{l+1} \pi^*_e S_{k,e} \subset F_{l} \pi^*_e S_{k,e} \subset \cdots \subset F_1 \pi^*_e S_{k,e} \subset F_0 \pi^*_e S_{k,e} = \pi^*_e S_{k,e}$ such that

$$F_i \pi^*_e S_{k,e} / F_{i+1} \pi^*_e S_{k,e} \cong S_{l,(-1)^i} \otimes \Lambda^i (U_l / U_k).$$  

In particular, we have the following short exact sequence on $\mathcal{G}_{k-1,k}$:

$$0 \rightarrow \pi^*_e S_{k-1,-e} \otimes \mathcal{O}_{\mathcal{G}_{k-1,k}}(1,-1) \rightarrow \pi^*_e S_{k-1,e} \rightarrow \pi^*_e S_{k,e} \rightarrow 0.$$  

**Corollary 6.5.** We have

$$\operatorname{rank}(S_{k,e}) = \begin{cases} 2^{m-k} & \text{if } n = 2m + 1, \\ 2^{m-k-1} & \text{if } n = 2m, \end{cases} \quad \operatorname{det} S_{k,e} \cong \begin{cases} \mathcal{O}_{\mathcal{G}_k}(2^{m-k-1}) & \text{if } n = 2m + 1, \\ \mathcal{O}_{\mathcal{G}_k}(2^{m-k-2}) & \text{if } n = 2m. \end{cases}$$

**Proof.** For $k = 0$ the claim is evident. For $k > 0$ we proceed by induction using (21). \hfill \qedsymbol

**Proposition 6.6.** The spinor bundles are (mutually) self-dual up to a twist: $S^*_e \cong S_{k,e}(-1)$.

**Proof.** We have seen that the (half)-spinor modules over the Clifford algebras are (mutually) self-dual and the duality isomorphisms (compatible with the module structures) are unique; hence the local isomorphisms glue into an isomorphism $S^*_e \cong S_{k,(-1)^m} \otimes L$ for some line bundle $L$ on $\mathcal{G}_k$. However, comparing determinants and using Corollary 6.5, we easily deduce that $L \cong \mathcal{O}_{\mathcal{G}_k}(-1)$. \hfill \qedsymbol

**Proposition 6.7.** There is a filtration on the bundle $\bigoplus_e S_e \otimes S^*_{k,e}$:

$$0 = F_{[(n-k)/2]+1} \left( \bigoplus_e S_e \otimes S^*_{k,e} \right) \subset F_{[(n-k)/2]} \left( \bigoplus_e S_e \otimes S^*_{k,e} \right) \subset \cdots$$  

$$\subset F_1 \left( \bigoplus_e S_e \otimes S^*_{k,e} \right) \subset F_0 \left( \bigoplus_e S_e \otimes S^*_{k,e} \right) = \bigoplus_e S_e \otimes S^*_{k,e},$$

such that $F_s(\bigoplus_e S_e \otimes S^*_{k,e}) / F_{s+1}(\bigoplus_e S_e \otimes S^*_{k,e}) \cong \Lambda^s U_k^1$. 


Note that the map $\pi$ is a projectivization of a vector bundle, and hence the functor $\pi_*$ is fully faithful, that is, we can (and will) compute $\text{RHom}(\pi_k S_\epsilon, \pi_k S_{\epsilon'})$ instead. Now look at the map $\pi_{k-1}$. It is a fibration in quadrics (of dimension $n - 2(k - 1) - 2 = n - 2k$), and it is clear that the restriction of the spinor bundle $S_{k-1,\epsilon}$ to any fiber of $\pi_{k-1}$ is isomorphic to the usual spinor bundle on this fiber. Therefore on any fiber of $\pi_{k-1}$ the bundle $S_{k-1,\epsilon}^* \otimes S_{k-1,\epsilon'}$ has no cohomology if $\epsilon \neq \epsilon'$ and has one-dimensional cohomology in degree 0 if $\epsilon = \epsilon'$. Since $\pi_{k-1}$ is flat, it follows that

$$\pi_{k-1*}(S_{k-1,\epsilon}^* \otimes S_{k-1,\epsilon'}) = \begin{cases} 0 & \text{if } \epsilon \neq \epsilon', \\ \text{a line bundle} & \text{if } \epsilon = \epsilon'. \end{cases}$$

On the other hand, if $\epsilon = \epsilon'$, then the bundle $S_{k-1,\epsilon}^* \otimes S_{k-1,\epsilon'}$ contains $\mathcal{O}_{\mathbb{F}_{k-1,\epsilon}}$ as a direct summand; hence its push-forward to $\mathbb{F}_{k-1}$ contains $\pi_{k-1*}(\mathcal{O}_{\mathbb{F}_{k-1,\epsilon}}) \cong \mathcal{O}_{\mathbb{F}_{k-1}}$ as a direct summand; hence the line bundle discussed above is trivial. Summarizing, we have

$$\pi_{k-1*}(S_{k-1,\epsilon}^* \otimes S_{k-1,\epsilon'}) = \begin{cases} 0 & \text{if } \epsilon \neq \epsilon', \\ \mathcal{O}_{\mathbb{F}_{k-1}} & \text{if } \epsilon = \epsilon'. \end{cases}$$

Similarly, for $0 \leq p \leq n - 2k - 1$ the bundle $S_{k-1,\epsilon}^*(0, -p)$ has no cohomology on any fiber of $\pi_{k-1}$, and hence

$$\pi_{k-1*}(S_{k-1,\epsilon}^*(0, -p)) = 0 \quad \text{for } 0 \leq p \leq n - 2k - 1 \text{ and any } \epsilon.$$  \hfill (22)

Now consider sequence (21) tensored by $S_{k-1,\epsilon}^*(0, -p)$:

$$0 \to S_{k-1,\epsilon}^* \otimes S_{k-1,\epsilon'}(1, -p - 1) \to S_{k-1,\epsilon}^*(0, -p) \otimes \pi_{k-1*}S_{k-1,\epsilon'} \to S_{k-1,\epsilon}^* \otimes S_{k-1,\epsilon'}(0, -p) \to 0.$$
Taking the push-forward to $\mathfrak{F}_{k-1}$ and using (22) and the projection formula, we deduce that
\[
\pi_{k-1*}(S_{k-1,ke}^* \otimes S_{k-1,ke'}(0,-p)) \cong \pi_{k-1*}(S_{k-1,ke}^* \otimes S_{k-1,ke'}(0,-p)) \otimes O_{\mathfrak{F}_{k-1}}(-1)[-1],
\]
for $0 \leq p \leq n - 2k - 1$. It follows by induction that
\[
\pi_{k-1*}(S_{k-1,ke}^* \otimes S_{k-1,ke'}(0,-p)) = \begin{cases} 
0 & \text{if } \epsilon \neq (-1)p', 
O_{\mathfrak{F}_{k-1}}(-p) & \text{if } \epsilon = (-1)p',
\end{cases}
\]
for $0 \leq p \leq n - 2k$. Hence
\[
\text{RHom}(\pi_{k*}S_{k}(p), \pi_{k*}S_{k'}) = \begin{cases} 
0 & \text{if } \epsilon \neq \epsilon', 0 \leq p \leq n - 2k, \text{ or } \epsilon = \epsilon' \text{ and } 1 \leq p \leq n - 2k,
\text{if } \epsilon = \epsilon' \text{ and } p = 0,
\end{cases}
\]
and this is precisely what we need.

It remains to note that $\mathfrak{F}_{k-1}$ is a Fano variety with canonical bundle $O_{\mathfrak{F}_{k-1}}(k - n)$; therefore the bundles $O_{\mathfrak{F}_{k-1}}(-p)$ have no cohomology on $\mathfrak{F}_{k-1}$ for $1 \leq p \leq n - 2k$. Hence
\[
\text{RHom}(\pi_{k*}S_{k}(p), \pi_{k*}S_{k'}) = \begin{cases} 
0 & \text{if } \epsilon \neq \epsilon', 0 \leq p \leq n - 2k, \text{ or } \epsilon = \epsilon' \text{ and } 1 \leq p \leq n - 2k,
\text{if } \epsilon = \epsilon' \text{ and } p = 0,
\end{cases}
\]
and this is precisely what we need.

It remains to note that $S_{k}(1) \cong S_{k,-e}$ by Proposition 6.6, and so (22) implies the desired vanishing of cohomology of twists of $S_{k}$. 

### 7. Odd-dimensional orthogonal case

Consider the isotropic Grassmannian $X = \text{OGr}(2, W)$ of two-dimensional isotropic subspaces in an orthogonal vector space $W$ of dimension $n = 2m + 1$. Let $U$ denote the restriction of the tautological rank 2 sub-bundle from $\text{Gr}(2, W)$ to $X = \text{OGr}(2, W)$. Also, let $S = S_2$ denote the spinor bundle on $X$ constructed in the previous section.

Consider the Lefschetz collection (4) on $\text{OGr}(2, W)$. This is a Lefschetz collection with the first block
\[
(O_X, U^*, S^2 U^*, \ldots, S^{m-2} U^*, S) \tag{23}
\]
and with the support partition
\[
\lambda^0 = (m, m, \ldots, m). \tag{24}
\]
In other words, the collection consists of vector bundles $S^l U^*(k)$ with integers $(k, l)$ from the set
\[
\Upsilon^0_{2m+1} = \{(k, l) \in \mathbb{Z}^2 \mid 0 \leq k \leq 2m - 3, 0 \leq l \leq m - 2\}. \tag{25}
\]
and of an additional collection
\[
(S, S(1), \ldots, S(2m - 3)) \tag{26}
\]
of twists of the spinor bundle. In Figure 5, we draw the regions $\Upsilon_{2m+1}$ and $\Upsilon^0_{2m+1}$ in the $(k, l)$-plane.

![Figure 5. Regions $\Upsilon^0_{2m+1}$](image)
THEOREM 7.1. Let $\Upsilon_0^{2m+1}$ be the set defined in (25). Then the Lefschetz collection \( \{ S'U^*(k) \mid (k,l) \in \Upsilon_0^{2m+1} \} \cup \{ S(k) \mid 0 \leq k \leq 2m-3 \} \) is a full exceptional collection in $D^b(O\text{Gr}(2,2m+1))$.

First of all we must check that the collection is exceptional.

LEMMA 7.2. The Lefschetz collection \( \{ S'U^*(k) \mid (k,l) \in \Upsilon_0^{2m+1} \} \cup \{ S(k) \mid 0 \leq k \leq 2m-3 \} \) is exceptional.

Proof. First, let us check that the collection \( \{ S'U^*(k) \mid (k,l) \in \Upsilon_0^{2m+1} \} \) is exceptional. The arguments of Lemmas 2.2 and 3.5 show that it suffices to check that $H^*(X, \Sigma^{p-k,-l-k}U^*) = 0$ for $(k,l) \in \Upsilon_0^{2m+1}$ and $0 \leq p \leq m-2$ (with an additional restriction $p < l$ for $k = 0$), and that $H^*(X, O_X) = k$.

For this we note that $X \subset \text{Gr}(2,W)$ is the zero locus of a regular section of the bundle $S^2U^*$ (given by the quadratic form on $W$). Therefore, on $\text{Gr}(2,W)$ we have the following Koszul resolution:

$$0 \to O(-3) \to S^2U(-1) \to S^2U \to O \to i_*O_X \to 0.$$  

Tensoring it by $\Sigma^{p-k,-l-k}U^*$ we obtain a resolution of $\Sigma^{p-k,-l-k}U^* \otimes i_*O_X \cong i_*(\Sigma^{p-k,-l-k}U^*)$ on $\text{Gr}(2,W)$. It is easy to see that

$$\Sigma^{p-k,-l-k}U^* \otimes S^2U \cong \Sigma^{p-k,-l-k-2}U^* \oplus \Sigma^{p-k-1,-l-k-1}U^* \oplus [\Sigma^{p-k-2,-l-k}U^*],$$

$$\Sigma^{p-k,-l-k}U^* \otimes S^2U(-1) \cong \Sigma^{p-k-1,-l-k-3}U^* \oplus \Sigma^{p-k-2,-l-k-2}U^* \oplus [\Sigma^{p-k-3,-l-k}U^*],$$

$$\Sigma^{p-k,-l-k}U^* \otimes O(-3) \cong \Sigma^{p-k-3,-l-k-3}U^*$$

(the terms in the brackets should be omitted if $p + l \leq 1$). All summands in the RHS can be rewritten in the form $\Sigma^{p-k',-l'-k'}U^*$ with $(k',l') \in \Upsilon_{2m+1}$ and $0 \leq p \leq m-1$ (with an additional restriction $p' < l'$ for $k' = 0$). As it was seen in Lemma 3.5, the cohomology of all these bundles on $\text{Gr}(2,W)$ vanishes, with the only exception $H^*(\text{Gr}(2,W), O_{\text{Gr}(2,W)}) = k$; therefore, the considered collection of vector bundles on $X$ is indeed exceptional.

Further, we note that the collection $(S, S(1), \ldots, S(2m-3))$ is exceptional by Proposition 6.8. It remains to check semi-orthogonality between $S'U^*(k)$ and $S(l)$.

Now let us check that $R\text{Hom}(S'U^*(k), S) = H^*(X, S \otimes S'U^*(-k)) = 0$ for all $(k,l) \in \Upsilon_0^{2m+1}$ with $k \geq 1$, that is, for all $1 \leq k \leq 2m-3$, $0 \leq l \leq 2m-2$. Actually, we will check that

$$H^*(X, S \otimes S'U^*(-k)) = 0 \quad \text{for all } 0 \leq l \leq m-2, \quad 1 \leq k \leq 2m-2.$$  

(27)

The proof is inductive in $l$. The case $l = 0$ is already proved in Proposition 6.8. Now consider the complex (20) tensored by $S^{l-1}U^*(-k)$:

$$0 \to S \otimes S^{l-1}U^*(-k-1) \to S \otimes S^{l-1}U^*(-k) \to S \otimes S^{l-1}U^*(-k) \to 0$$

By Proposition 6.3, its cohomology is isomorphic to $S \otimes U \otimes S^{l-1}U^*(-k)$. Consider the hypercohomology spectral sequence of this complex. By the induction hypothesis and the exceptionality of the collection \( \{ S'U^*(k) \mid (k,l) \in \Upsilon_0^{2m+1} \} \) proved above, the cohomology of all terms of this complex vanish if $2 - l \leq k \leq 2m-3$. Therefore, the cohomology of the bundle $S \otimes U \otimes S^{l-1}U^*(-k)$ vanishes as well. However, $S \otimes S'U^*(-k)$ is a direct summand of $S \otimes U \otimes S^{l-1}U^*(-k)$, and hence its cohomology vanishes as well. It remains to consider only the bundle $S \otimes S'U^*(-k)$ with $k = 2m-2$ and $k = 1-l$ which are not covered by the above arguments. Let us start with $k = 1-l$. Note that $S^{l-1}U(l-1) \cong S^{l-1}U^*$; so the above complex
takes the form
\[ 0 \rightarrow S \otimes S^{l-1}U'(-1) \rightarrow S \otimes S^{l-1}U' \rightarrow S \otimes S^{l-1}U^* \rightarrow 0, \]
and its cohomology is isomorphic to \( S \otimes U \otimes S^{l-1}U^* \cong S \otimes S^{l-2}U^* \otimes S \otimes S^lU^*(-1) \). The cohomology of the first term of the complex vanishes by the induction hypothesis, while for the middle and the last term of the complex the Borel–Bott–Weil theorem gives
\[ H^\bullet(X, S \otimes S^{l-1}U^*) \cong V(\omega_m) \otimes V((l-1)\omega_1), \quad H^\bullet(X, S \otimes S^{l-1}U^*) \cong V(\omega_m + (l-1)\omega_1) \]
(the application of the Borel–Bott–Weil theorem in this case is absolutely straightforward since the corresponding weights are dominant), and the map between them induced by the differential of the complex is the canonical projection. However, according to the representation theory of the Lie group \( O(2m+1) \) we have \( V(\omega_m) \otimes V((l-1)\omega_1) \cong V(\omega_m + (l-1)\omega_1) \oplus V(\omega_m + (l-2)\omega_1) \); hence
\[ H^\bullet(X, S \otimes S^{l-2}U^* \oplus S \otimes S^lU^*(-1)) \cong V(\omega_m + (l-2)\omega_1). \]
It remains to note that \( H^\bullet(X, S \otimes S^{l-2}U^*) \cong V(\omega_m + (l-2)\omega_1) \) again by Borel–Bott–Weil. Hence the cohomology of \( S \otimes S^lU^*(-1) \) vanishes, but this is precisely what we need.

Finally, the case \( k = 2m - 2 \) follows by Serre duality. The canonical bundle of \( X \) is isomorphic to \( \mathcal{O}_X(2-2m) \); hence
\[ H^{\dim X-\bullet}(X, S \otimes S^lU(2-2m)) \cong H^\bullet(X, S^* \otimes S^lU^*)^*, \]
but by Proposition 6.6 \( S^* \cong S(-1) \), while \( S^lU^* \cong S^lU(l-1) \); hence
\[ S^* \otimes S^lU^* \cong S \otimes S^lU(l-1). \]

Finally we note that by Serre duality we have \( \text{RHom}(S(k), S^lU^*) \cong \text{RHom}(S^lU^*, S(k + 2 - 2m)) = H^\bullet(X, S \otimes S^lU(k + 2 - 2m)) \) which is zero for \( 0 \leq k \leq 2m - 3 \) by (27).

\[ \square \]

It remains to prove the fullness of the collection. We use the same method as in Sections 4 and 5 with the induction step that changes \( \dim W \) by 2.

Consider the set
\[ \bar{\Upsilon}_{2m-1}^O = \{(k, l) \in \mathbb{Z}^2 \mid 0 \leq k \leq 2m - 3, 0 \leq l \leq m - 1\}. \tag{28} \]
In Figure 6, the region \( \Upsilon_{2m+1}^O \) is drawn together with the region \( \bar{\Upsilon}_{2m-1}^O \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{Regions \( \bar{\Upsilon}_{2m-1}^O \)}
\end{figure}

**Lemma 7.3.** For any \( (k, l) \in \bar{\Upsilon}_{2m-1}^O \) the vector bundle \( S^lU^*(k) \) lies in the triangulated category generated by the Lefschetz collection \( \{ S^lU^*(k) \mid (k, l) \in \Upsilon_{2m+1}^O \} \cup \{ S(k) \mid 0 \leq k \leq 2m - 3 \} \) in \( D^b(\mathcal{OGr}(2, W)) \).
Proof. Let \( \mathcal{D} \) denote the triangulated subcategory of \( \mathcal{D}^b(\text{OGr}(2,W)) \) generated by the Lefschetz collection \( \{ S^l U^*(k) \mid (k,l) \in \mathcal{Y}_2 \} \cup \{ S(k) \mid 0 \leq k \leq 2m - 3 \} \). Note that

\[
\mathcal{Y}_2 \setminus \mathcal{Y}_2 \setminus \{ (0, m - 1), (1, m - 1), \ldots , (2m - 3, m - 1) \}.
\]

Therefore, we have to check that the vector bundles

\[
S^{m-1} U^*, S^{m-1} U^*(1), \ldots , S^{m-1} U^*(2m - 3)
\]

lie in \( \mathcal{D} \). Consider the filtration on the sheaf \( S^* \otimes S \), dual to the filtration described in Proposition 6.7. The quotients of this filtration take the form \( \Lambda^{2s}(W/U) \). Note that if \( 2s < m - 1 \) then \( \Lambda^{2s}(W/U) \) admits a resolution (10), which shows that we have \( \Lambda^{2s}(W/U) \in (\mathcal{O}_X, U^*(1), S^2 U^*(2), \ldots , S^{2s} U^*(-2s)) \). Hence

\[
\Lambda^{2s}(W/U)(t) \in \mathcal{D} \quad \text{if } 2s < m - 1 \text{ and } 2s \leq t \leq 2m - 3.
\] (29)

Similarly, if \( 2s > m \) then \( \Lambda^{2s}(W/U) \cong \Lambda^{2m-1-2s} U^1 \) admits a resolution (9), which shows that we have \( \Lambda^{2s}(W/U) \in (\mathcal{O}_X(1), U^*(1), S^2 U^*(1), \ldots , S^{2m-1-2s} U^*(-1)) \); hence

\[
\Lambda^{2s}(W/U)(t) \in \mathcal{D} \quad \text{if } 2s > m \text{ and } -1 \leq t \leq 2m - 4.
\] (30)

However, there exists precisely one even integer \( 2s \) which is not covered by the previous two cases. It is either \( m - 1 \) if \( m \) is odd, or \( m \) if \( m \) is even. We are forced to consider these two cases separately.

First assume that \( m \) is odd. Then the inclusions (29), (30) together with

\[
S^* \otimes S(t) \in \mathcal{D} \quad \text{if } 0 \leq t \leq 2m - 3.
\] (31)

imply that for \( m - 3 \leq t \leq 2m - 4 \) both \( S^* \otimes S(t) \) and all quotients of its filtration except for \( \Lambda^{m-1}(W/U)(t) \) lie in \( \mathcal{D} \). However, since \( \mathcal{D} \) is triangulated it follows that \( \Lambda^{m-1}(W/U)(t) \in \mathcal{D} \) for \( m - 3 \leq t \leq 2m - 4 \) as well; but we see that resolution (10) implies that \( S^{m-1} U^*(t - m + 1) \in \mathcal{D} \) for \( m - 2 \leq t \leq 2m - 4 \). Thus we deduce that for odd \( m \) we have

\[
S^{m-1} U^*(-1), S^{m-1} U^*, \ldots , S^{m-1} U^*(m - 3) \in \mathcal{D}.
\] (32)

Now assume that \( m \) is even. Then the inclusions (29)–(31) imply that for \( m - 2 \leq t \leq 2m - 4 \) both \( S^* \otimes S(t) \) and all quotients of its filtration except for \( \Lambda^m(W/U)(t) \) lie in \( \mathcal{D} \). However, since \( \mathcal{D} \) is triangulated it follows that \( \Lambda^m(W/U)(t) \cong \Lambda^m U^1 / \mathcal{L} \) for \( m - 2 \leq t \leq 2m - 4 \) as well; but we see that resolution (9) implies that \( S^{m-1} U^*(t + 1) \in \mathcal{D} \) for \( m - 2 \leq t \leq 2m - 4 \). Thus we deduce that for even \( m \) we have

\[
S^{m-1} U^*(m - 1), S^{m-1} U^*(m), \ldots , S^{m-1} U^*(2m - 3) \in \mathcal{D}.
\] (33)

Finally, we are going to use the following trick to show that the other twists of \( S^{m-1} U^* \) also lie in \( \mathcal{D} \). Consider the vector bundle \( U^1 / \mathcal{L} \) on \( X \). It comes with a non-degenerate quadratic form induced by the form on \( W \). Its rank is \( 2m + 1 - 4 = 2m - 3 \). It follows that for any \( t \) we have an isomorphism

\[
\Lambda^{m-1}(U^1 / \mathcal{L})(t) \cong \Lambda^{m-2}(U^1 / \mathcal{L})(t).
\] (34)
Now note that $\mathcal{U}^\perp/\mathcal{U}$ is the cohomology of the complex $\mathcal{U} \to W \otimes \mathcal{O}_X \to \mathcal{U}^\ast$. Taking exterior powers we deduce that

$$
\Lambda^\ast(\mathcal{U}^\perp/\mathcal{U}) \cong \left\{ \begin{array}{c}
S^t \mathcal{U} \to W \otimes S^{t-1} \mathcal{U} \to \cdots \to \Lambda^{s-2} W \otimes S^2 \mathcal{U} \to \Lambda^{s-1} W \otimes \mathcal{U} \to \Lambda^s W \otimes \mathcal{O}_X \\
U^\ast \otimes S^{t-1} \mathcal{U} \to \cdots \to U^\ast \otimes \Lambda^{s-2} W \otimes S^2 \mathcal{U} \to U^\ast \otimes \Lambda^{s-1} W \otimes \mathcal{U} \to U^\ast \otimes \Lambda^s W \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
S^{s-2} U^\ast \otimes S^2 \mathcal{U} \to S^{s-2} U^\ast \otimes W \otimes \mathcal{U} \to S^{s-2} U^\ast \otimes \Lambda^s W \\
S^{s-1} U^\ast \otimes \mathcal{U} \to S^{s-1} U^\ast \otimes W \\
S^s \mathcal{U}^\ast
\end{array} \right\}
$$

Taking into account isomorphisms

$$
S^a \mathcal{U}^\ast \otimes S^b \mathcal{U} \cong \bigoplus_{c=0}^{\min\{a,b\}} S^{a+b-2c} \mathcal{U}^\ast (c-b),
$$

we see that the isomorphism (34) with $t = m - 2$ can be considered as an expression for $S^{m-1} \mathcal{U}^\ast (m-2)$ in terms of $S^m \mathcal{U}^\ast (-1), \ldots, S^{m-1} \mathcal{U}^\ast (m-3)$ and other objects of $\mathcal{D}$. Taking into account (32) we deduce that $S^{m-1} \mathcal{U}^\ast (m-2) \in \mathcal{D}$ for odd $m$. Repeating the same argument for the isomorphisms (34) with $t = m-1, m, \ldots, 2m-3$ we deduce that $S^{m-1} \mathcal{U}^\ast (t) \in \mathcal{D}$ for these $t$ for odd $m$. Similarly, if $m$ is even then we see that the isomorphism (34) with $t = 2m-3$ can be considered as an expression for $S^{m-1} \mathcal{U}^\ast (m-2)$ in terms of $S^m \mathcal{U}^\ast (m-1), \ldots, S^{m-1} \mathcal{U}^\ast (2m-3)$ and other objects of $\mathcal{D}$. Taking into account (33) we deduce that $S^{m-1} \mathcal{U}^\ast (m-2) \in \mathcal{D}$ for even $m$. Repeating the same argument for the isomorphisms (34) with $t = 2m-4, 2m-5, \ldots, m-1$, we deduce that $S^{m-1} \mathcal{U}^\ast (t+1-m) \in \mathcal{D}$ for these $t$ for even $m$. In both cases we see that all desired twists of $S^{m-1} \mathcal{U}^\ast$ lie in $\mathcal{D}$. 

We will also need the following lemma.

**Lemma 7.4.** The bundles $S \otimes \mathcal{U}^\ast (k)$ for $0 \leq k \leq 2m-4$ as well as the bundles $S \otimes S^2 \mathcal{U}^\ast (k)$ for $0 \leq k \leq 2m-5$ lie in the triangulated category generated by the Lefschetz collection \{ $S^l \mathcal{U}^\ast (k) \mid (k,l) \in \mathcal{Y}^0_{2m+1}$\} $\cup$ \{ $S^l \mathcal{U}^\ast (k) \mid 0 \leq k \leq 2m-3$\} in $\mathcal{D}^b(\mathcal{OGr}(2,W))$.

**Proof.** Consider the complex

$$
0 \to S(-1) \to S \otimes \mathcal{O}_X \to S \to 0.
$$

By Proposition 6.3 it is quasi-isomorphic to $S \otimes \mathcal{U} \cong S \otimes \mathcal{U}^\ast (-1)$. Twisting it by $1, 2, \ldots, 2m-3$ we deduce that $S \otimes \mathcal{U}^\ast (k)$ lies in the desired triangulated category for $0 \leq k \leq 2m-4$. Further, consider the same complex tensored by $\mathcal{U}^\ast (-1):

$$
0 \to S \otimes \mathcal{U}^\ast (-2) \to S \otimes \mathcal{U}^\ast (-1) \to S \otimes \mathcal{U}^\ast (-1) \to 0.
$$

It is quasi-isomorphic to $S \otimes \mathcal{U} \otimes \mathcal{U}^\ast (-1) \cong S(-1) \oplus (S \otimes S^2 \mathcal{U}^\ast (-2))$. Twisting it by $2, 3, \ldots, 2m-3$ we deduce that $S \otimes S^2 \mathcal{U}^\ast (k)$ lies in the desired triangulated category for $0 \leq k \leq 2m-5$. 

The last preparatory result is the following lemma.
Lemma 7.5. For any two-dimensional subspace $\langle w_1, w_2 \rangle \subset W \cong W^* = H^0(\text{OGr}(2, W), \mathcal{U}^*)$ such that the quadratic form $q$ is non-degenerate on $\langle w_1, w_2 \rangle$, the zero locus of the corresponding section $\phi = \phi_{w_1, w_2} \in H^0(\text{OGr}(2, W), \mathcal{U}^* \oplus \mathcal{U}^*)$ on $X = \text{OGr}(2, W)$ is the isotropic Grassmannian $X_{w_1, w_2} = \text{OGr}(2, \langle w_1, w_2 \rangle_{2}) \subset \text{OGr}(2, W) = X$. Moreover, we have the following resolution of the structure sheaf $\mathcal{O}_{X_{\phi}}$ on $X$:

$$0 \to \mathcal{O}_X(-2) \to \mathcal{U}(-1) \oplus \mathcal{U}(-1) \to \mathcal{O}_X(-1)^{\oplus 3} \oplus S^2 \mathcal{U} \to \mathcal{U} \oplus \mathcal{U} \to \mathcal{O}_X \to i_{\phi*} \mathcal{O}_{X_{w_1, w_2}} \to 0, \quad (35)$$

where $i_{\phi*} : X_{w_1, w_2} \to X$ is the embedding.

**Proof.** The first part is evident. For the second part we note that any such section $\phi = \phi_{w_1, w_2}$ of $\mathcal{U}^* \oplus \mathcal{U}^*$ is regular since dim $X_{w_1, w_2} = 2(n - 2) = 7 = \text{dim} X - 4$, and so the sheaf $i_{\phi*} \mathcal{O}_{X_{w_1, w_2}}$ admits a Koszul resolution which takes the form $(16)$. \qed

Now we are ready for the proof of the theorem. We use induction on $m$. The base of induction, $m = 2$, is clear. Indeed, in this case $X = \text{OGr}(2, W) = \mathbb{P}^3$ and the Lefschetz collection takes the form $(\mathcal{O}_{p_3}, \mathcal{O}_{p_3}(1), \mathcal{O}_{p_3}(2), \mathcal{O}_{p_3}(3))$ (since $\mathcal{O}_X(1) = \mathcal{O}_{p_3}(2)$ and $S = \mathcal{O}_{p_3}(1)$) which is well known to be full.

Now assume that the fullness of the corresponding Lefschetz collection is already proved for $m - 1$. Also, assume that the Lefschetz collection for $m$ is not full. Then by [2, Theorem 3.2(a) and Lemma 3.1], there exists an object $F \in D^b(X)$, right orthogonal to all bundles in the collection. Then by Lemmas 7.3 and 7.4 we have

$$H^*(X, S^l \mathcal{U}(-k) \otimes F) = 0 \quad \text{for all } (k, l) \in \overline{\mathcal{Y}}_{2m-2}^Q$$

$$H^*(X, S^*(-k) \otimes F) = 0 \quad \text{for all } 0 \leq k \leq 2m - 3$$

$$H^*(X, S^* \otimes \mathcal{U}(-k) \otimes F) = 0 \quad \text{for all } 0 \leq k \leq 2m - 4$$

$$H^*(X, S^* \otimes S^2 \mathcal{U}(-k) \otimes F) = 0 \quad \text{for all } 0 \leq k \leq 2m - 5$$

Let us check that $i_{\phi*} F = 0$ for any $\phi = \phi_{w_1, w_2}$, as in Lemma 7.5. The same argument as in Section 5 shows that $i_{\phi*} F$ lies in the right orthogonal to the subcategory of $D^b(X_{\phi})$ generated by the exceptional collection $\{ S^l \mathcal{U}^*(k) \mid (k, l) \in \mathcal{Y}_{2m-1}^Q \}$. Similarly, tensor the resolution $(35)$ by $S^*(-k) \otimes F$ for $0 \leq k \leq 2m - 5$. It follows that the cohomology on $X$ of the first five terms of the above complexes vanishes. Therefore we have

$$R\text{Hom}_{X_{\phi}}(S(k), i_{\phi*} F) = H^*(X_{\phi}, S^*(-k) \otimes i_{\phi*} F) = 0 \quad \text{for } 0 \leq k \leq 2m - 5.$$

Thus $i_{\phi*} F$ lies in the right orthogonal to the subcategory of $D^b(X_{\phi})$ generated by the exceptional collection $\{ S(k) \mid 0 \leq k \leq 2m - 5 \}$. However, by the induction hypothesis the collection $\{ S^l \mathcal{U}^*(k) \mid (k, l) \in \overline{\mathcal{Y}}_{2m-1}^Q \} \cup \{ S(k) \mid 0 \leq k \leq 2m - 5 \}$ on $X_{\phi}$ is full, and hence $i_{\phi*} F = 0$. Therefore we conclude by the following lemma.

**Lemma 7.6.** If for $F \in D^b(X)$ we have $i_{\phi*} F = 0$ for any two-dimensional subspace $\langle w_1, w_2 \rangle \subset W$ such that the quadratic form $q$ is non-degenerate on $\langle w_1, w_2 \rangle$ then $F = 0$.

**Proof.** Assume that $F \neq 0$. Let $q$ be the maximal integer such that $\mathcal{H}^q(F) \neq 0$; take a point $x \in \text{supp} \mathcal{H}^q(F)$ and choose $\langle w_1, w_2 \rangle \subset W$ such that $x \in X_{\phi}$ (this is equivalent to the orthogonality of $w_1$ and $w_2$ with the two-dimensional subspace of $W$ corresponding to $x \in X = \text{OGr}(2, W)$). Since the functor $i_{\phi*}$ is left-exact it easily follows that $\mathcal{H}^q(i_{\phi*} F) \neq 0$, and so $i_{\phi*} F \neq 0$. \qed
Thus we have proved that the desired collection is indeed full.

**Remark 7.7.** The same arguments allow to construct an exceptional collection on OGr(2, W) in the case of even $n = \dim W$. Explicitly, if $n = 2m$, then one can show that the Lefschetz collection

$$
\begin{pmatrix}
\mathcal{S}_- & \mathcal{S}_-(1) & \cdots & \mathcal{S}_-(m-1) & \mathcal{S}_-(m-1) & \cdots & \mathcal{S}_-(2m-4) \\
\mathcal{S}_+ & \mathcal{S}_+(1) & \cdots & \mathcal{S}_+(m-1) & \mathcal{S}_+(m-1) & \cdots & \mathcal{S}_+(2m-4) \\
\mathcal{S}_{m-2}\mathcal{U}^* & \mathcal{S}_{m-2}\mathcal{U}^*(1) & \cdots & \mathcal{S}_{m-2}\mathcal{U}^*(m-2) \\
\mathcal{S}_{m-3}\mathcal{U}^* & \mathcal{S}_{m-3}\mathcal{U}^*(1) & \cdots & \mathcal{S}_{m-3}\mathcal{U}^*(m-2) & \mathcal{S}_{m-3}\mathcal{U}^*(m-1) & \cdots & \mathcal{S}_{m-3}\mathcal{U}^*(2m-4) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\mathcal{U}^* & \mathcal{U}^*(1) & \cdots & \mathcal{U}^*(m-2) & \mathcal{U}^*(m-1) & \cdots & \mathcal{U}^*(2m-4) \\
\mathcal{O} & \mathcal{O}(1) & \cdots & \mathcal{O}(m-2) & \mathcal{O}(m-1) & \cdots & \mathcal{O}(2m-4)
\end{pmatrix}
$$

is exceptional. However, this collection is definitely not full (its length is by 1 less than the rank of the Grothendieck group $K_0(OGr(2, W))$). Therefore, to obtain a full collection one should add one more exceptional object to the first block $(\mathcal{O}, \mathcal{U}^*, \ldots, \mathcal{S}_{m-2}\mathcal{U}^*, \mathcal{S}_+, \mathcal{S}_-)$ of the above collection. This indeed can be done, but the object in question turns out to be a complex (not a pure sheaf), and so the picture became more complicated. This is not quite satisfactory and it seems that there should exist a better full exceptional collection.

**References**


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