

# Semiorthogonal decompositions in algebraic geometry

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**Abstract.** In this review we discuss what is known about semiorthogonal decompositions of derived categories of algebraic varieties. We review existing constructions, especially the homological projective duality approach, and discuss some related issues such as categorical resolutions of singularities.

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## 1. Introduction

In recent years an extensive investigation of semiorthogonal decompositions of derived categories of coherent sheaves on algebraic varieties has been done, and now we know quite a lot of examples and some general constructions. With time it is becoming more and more clear that semiorthogonal components of derived categories can be thought of as the main objects in noncommutative algebraic geometry. In this paper I will try to review what is known in this direction — how one can construct semiorthogonal decompositions and how one can use them.

In section 2 we will recall the basic notions, discuss the most frequently used semiorthogonal decompositions, and state the base change formula. In section 3 we review the theory of homological projective duality which up to now is the most powerful method to construct semiorthogonal decompositions. In section 4 we discuss categorical resolutions of singularities, a subject interesting by itself, and at the same time inseparable from homological projective duality. In section 5 examples of homologically projectively dual varieties are listed. Finally, in section 6 we discuss semiorthogonal decompositions of varieties of small dimension.

I should stress that in the area of algebraic geometry described in this paper there are more questions than answers, but it really looks very promising. Also, due to volume constraints I had to leave out many interesting topics closely related to the main subject, such as the categorical Griffiths component, Hochschild homology and cohomology, and many others.

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## 2. Semiorthogonal decompositions

This paper can be considered as a continuation and a development of the ICM 2002 talk [7] of Alexei Bondal and Dmitri Orlov. So I will freely use results and definitions from [7] and restrict myself to a very short reminder of the most basic notion. In particular, the reader is referred to [7] for the definition of a Serre functor, Fourier–Mukai transform, etc.

**2.1. A short reminder.** Recall that a semiorthogonal decomposition of a triangulated category  $\mathcal{T}$  is a collection  $\mathcal{A}_1, \dots, \mathcal{A}_n$  of full triangulated subcategories such that:

- (a) for all  $1 \leq j < i \leq n$  and any objects  $A_i \in \mathcal{A}_i, A_j \in \mathcal{A}_j$  one has  $\text{Hom}_{\mathcal{T}}(A_i, A_j) = 0$ ;
- (b) the smallest triangulated subcategory of  $\mathcal{T}$  containing  $\mathcal{A}_1, \dots, \mathcal{A}_n$  coincides with  $\mathcal{T}$ .

We will use the notation  $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$  for a semiorthogonal decomposition of  $\mathcal{T}$  with components  $\mathcal{A}_1, \dots, \mathcal{A}_n$ .

We will be mostly interested in semiorthogonal decompositions of  $\mathbf{D}^b(\text{coh}(X))$ , the *bounded derived category of coherent sheaves* on an algebraic variety  $X$  which in most cases will be assumed to be smooth and projective over a base field  $k$ .

Recall that a full triangulated subcategory  $\mathcal{A} \subset \mathcal{T}$  is *admissible* if its embedding functor  $i : \mathcal{A} \rightarrow \mathcal{T}$  has both left and right adjoint functors  $i^*, i^! : \mathcal{T} \rightarrow \mathcal{A}$ . An admissible subcategory  $\mathcal{A} \subset \mathcal{T}$  gives rise to a pair of semiorthogonal decompositions

$$\mathcal{T} = \langle \mathcal{A}, {}^\perp \mathcal{A} \rangle \quad \text{and} \quad \mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle, \quad (2.1)$$

where

$${}^\perp \mathcal{A} := \{T \in \mathcal{T} \mid \text{Hom}(T, A[t]) = 0 \text{ for all } A \in \mathcal{A}, t \in \mathbb{Z}\}, \quad (2.2)$$

$$\mathcal{A}^\perp := \{T \in \mathcal{T} \mid \text{Hom}(A[t], T) = 0 \text{ for all } A \in \mathcal{A}, t \in \mathbb{Z}\}, \quad (2.3)$$

are the left and the right orthogonals to  $\mathcal{A}$  in  $\mathcal{T}$ . More generally, if  $\mathcal{A}_1, \dots, \mathcal{A}_m$  is a semiorthogonal collection of admissible subcategories in  $\mathcal{T}$ , then for each  $0 \leq k \leq m$  there is a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_k, {}^\perp \langle \mathcal{A}_1, \dots, \mathcal{A}_k \rangle \cap \langle \mathcal{A}_{k+1}, \dots, \mathcal{A}_m \rangle^\perp, \mathcal{A}_{k+1}, \dots, \mathcal{A}_m \rangle. \quad (2.4)$$

The simplest example of an admissible subcategory is the one generated by an exceptional object. Recall that an object  $E$  is *exceptional* if one has  $\text{Hom}(E, E) = k$  and  $\text{Hom}(E, E[t]) = 0$  for  $t \neq 0$ . An *exceptional collection* is a collection of exceptional objects  $E_1, E_2, \dots, E_m$  such that  $\text{Hom}(E_i, E_j[t]) = 0$  for all  $i > j$  and all  $t \in \mathbb{Z}$ . An exceptional collection in  $\mathcal{T}$  gives rise to a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}, E_1, \dots, E_m \rangle \quad \text{with} \quad \mathcal{A} = \langle E_1, \dots, E_m \rangle^\perp. \quad (2.5)$$

Here  $E_i$  denotes the subcategory generated by the same named exceptional object. If the category  $\mathcal{A}$  in (2.5) is zero the exceptional collection is called *full*.

**2.2. Full exceptional collections.** There are several well-known and quite useful semiorthogonal decompositions. The simplest example is the following

**Theorem 2.1** (Beilinson’s collection). *There is a full exceptional collection*

$$\mathbf{D}^b(\text{coh}(\mathbb{P}^n)) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle. \quad (2.6)$$

Of course, twisting by  $\mathcal{O}_{\mathbb{P}^n}(t)$  we get  $\langle \mathcal{O}_{\mathbb{P}^n}(t), \mathcal{O}_{\mathbb{P}^n}(t + 1), \dots, \mathcal{O}_{\mathbb{P}^n}(t + n) \rangle$  which is also a full exceptional collection for each  $t \in \mathbb{Z}$ .

A bit more general is the Grassmannian variety:

**Theorem 2.2** (Kapranov’s collection, [19]). *Let  $\text{Gr}(k, n)$  be the Grassmannian of  $k$ -dimensional subspaces in a vector space of dimension  $n$ . Let  $\mathcal{U}$  be the tautological subbundle of rank  $k$ . If  $\text{char } k = 0$  then there is a semiorthogonal decomposition*

$$\mathbf{D}^b(\text{coh}(\text{Gr}(k, n))) = \langle \Sigma^\alpha \mathcal{U}^\vee \rangle_{\alpha \subset R(k, n-k)}, \tag{2.7}$$

where  $R(k, n-k)$  is the  $k \times (n-k)$  rectangle,  $\alpha$  is a Young diagram, and  $\Sigma^\alpha$  is the associated Schur functor.

When  $\text{char } k > 0$  there is an exceptional collection as well, but it is a bit more complicated, see [9].

Another interesting case is the case of a smooth quadric  $Q^n \subset \mathbb{P}^{n+1}$ .

**Theorem 2.3** (Kapranov’s collection, [19]). *When  $\text{char } k \neq 2$  and  $k$  is algebraically closed, there is a full exceptional collection*

$$\mathbf{D}^b(\text{coh}(Q^n)) = \begin{cases} \langle S, \mathcal{O}_{Q^n}, \mathcal{O}_{Q^n}(1), \dots, \mathcal{O}_{Q^n}(n-1) \rangle, & \text{if } n \text{ is odd} \\ \langle S^-, S^+, \mathcal{O}_{Q^n}, \mathcal{O}_{Q^n}(1), \dots, \mathcal{O}_{Q^n}(n-1) \rangle, & \text{if } n \text{ is even} \end{cases} \tag{2.8}$$

where  $S$  and  $S^\pm$  are the spinor bundles.

Many exceptional collections have been constructed on other rational homogeneous spaces, see e.g. [48], [32], [47], [42], [11], and [41]. Full exceptional collections on smooth toric varieties (and stacks) were constructed by Kawamata [20]. Also exceptional collections were constructed on del Pezzo surfaces [46], some Fano threefolds [24, 45] and many other varieties.

**2.3. Relative versions.** Let  $S$  be a scheme and  $E$  a vector bundle of rank  $r$  on it. Let  $\mathbb{P}_S(E)$  be its projectivization,  $f : \mathbb{P}_S(E) \rightarrow S$  the projection, and  $\mathcal{O}_{\mathbb{P}_S(E)/S}(1)$  the Grothendieck line bundle on  $\mathbb{P}_S(E)$ .

**Theorem 2.4** ([46]). *For each  $i \in \mathbb{Z}$  the functor*

$$\Phi_i : \mathbf{D}^b(\text{coh}(S)) \rightarrow \mathbf{D}^b(\text{coh}(\mathbb{P}_S(E))), \quad F \mapsto Lf^*(F) \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}_S(E)/S}(i) \tag{2.9}$$

is fully faithful, and there is a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(\mathbb{P}_S(E))) = \langle \Phi_0(\mathbf{D}^b(\text{coh}(S))), \dots, \Phi_{r-1}(\mathbf{D}^b(\text{coh}(S))) \rangle. \tag{2.10}$$

Of course, analogously to the case of a projective space, one can replace the sequence of functors  $\Phi_0, \dots, \Phi_{r-1}$  by  $\Phi_t, \dots, \Phi_{t+r-1}$  for any  $t \in \mathbb{Z}$ .

An interesting new feature appears for Severi–Brauer varieties. Recall that a Severi–Brauer variety over  $S$  is a morphism  $f : X \rightarrow S$  which étale locally is isomorphic to a projectivization of a vector bundle. A Severi–Brauer variety  $X$  can be constructed from a torsion element in the Brauer group  $\text{Br}(S)$  of  $S$ .

**Theorem 2.5** (Bernardara’s decomposition, [3]). *Let  $f : X \rightarrow S$  be a Severi–Brauer variety of relative dimension  $n$  and  $\beta \in \text{Br}(S)$  its Brauer class. Then for each  $i \in \mathbb{Z}$  there is a fully faithful functor  $\Phi_i : \mathbf{D}^b(\text{coh}(S, \beta^i)) \rightarrow \mathbf{D}^b(\text{coh}(X))$  and a semiorthogonal decomposition*

$$\mathbf{D}^b(\text{coh}(X)) = \langle \Phi_0(\mathbf{D}^b(\text{coh}(S))), \Phi_1(\mathbf{D}^b(\text{coh}(S, \beta))), \dots, \Phi_n(\mathbf{D}^b(\text{coh}(S, \beta^n))) \rangle. \quad (2.11)$$

Here  $\text{coh}(S, \beta^i)$  is the category of  $\beta^i$ -twisted coherent sheaves on  $S$  and the functor  $\Phi_i$  is given by  $F \mapsto Lf^*(F) \otimes^{\mathbb{L}} \mathcal{O}_{X/S}(i)$ , where the sheaf  $\mathcal{O}_{X/S}(i)$  is well defined as a  $f^*\beta^{-i}$ -twisted sheaf.

Another important semiorthogonal decomposition can be constructed for a smooth blow-up. Let  $Y \subset X$  be a locally complete intersection subscheme of codimension  $c$  and let  $\tilde{X}$  be the blowup of  $X$  with center in  $Y$ . Let  $f : \tilde{X} \rightarrow X$  be the blowup morphism and  $D \subset \tilde{X}$  the exceptional divisor of the blowup. Let  $i : D \rightarrow \tilde{X}$  be the embedding and  $p : D \rightarrow Y$  the natural projection (the restriction of  $f$  to  $D$ ). Note that  $D \cong \mathbb{P}_Y(\mathcal{N}_{Y/X})$  is the projectivization of the normal bundle.

**Theorem 2.6** (Blowup formula, [46]). *The functor  $Lf^* : \mathbf{D}^b(\text{coh}(X)) \rightarrow \mathbf{D}^b(\text{coh}(\tilde{X}))$  as well as the functors*

$$\Psi_k : \mathbf{D}^b(\text{coh}(Y)) \rightarrow \mathbf{D}^b(\text{coh}(\tilde{X})), \quad F \mapsto Ri_*(Lp^*(F) \otimes^{\mathbb{L}} \mathcal{O}_{D/Y}(k)),$$

are fully faithful for all  $k \in \mathbb{Z}$ , and there is a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(\tilde{X})) = \langle Lf^*(\mathbf{D}^b(\text{coh}(X))), \Psi_0(\mathbf{D}^b(\text{coh}(Y))), \dots, \Psi_{c-2}(\mathbf{D}^b(\text{coh}(Y))) \rangle. \quad (2.12)$$

Finally, consider a flat fibration in quadrics  $f : X \rightarrow S$ . In other words, assume that  $X \subset \mathbb{P}_S(E)$  is a divisor of relative degree 2 in a projectivization of a vector bundle  $E$  of rank  $n + 2$  on a scheme  $S$  corresponding to a line subbundle  $\mathcal{L} \subset S^2E^\vee$ .

**Theorem 2.7** (Quadratic fibration formula, [31]). *For each  $i \in \mathbb{Z}$  there is a fully faithful functor*

$$\Phi_i : \mathbf{D}^b(\text{coh}(S)) \rightarrow \mathbf{D}^b(\text{coh}(X)), \quad F \mapsto Lf^*(F) \otimes^{\mathbb{L}} \mathcal{O}_{X/S}(i)$$

and a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathbf{D}^b(\text{coh}(S, \mathcal{C}_0)), \Phi_0(\mathbf{D}^b(\text{coh}(S))), \dots, \Phi_{n-1}(\mathbf{D}^b(\text{coh}(S))) \rangle, \quad (2.13)$$

where  $\mathcal{C}_0$  is the sheaf of even parts of Clifford algebras on  $S$  associated with the quadric fibration  $X \rightarrow S$ .

The sheaf  $\mathcal{C}_0$  is a sheaf of  $\mathcal{O}_S$ -algebras which as an  $\mathcal{O}_S$ -module is isomorphic to

$$\mathcal{C}_0 \cong \mathcal{O}_S \oplus (\Lambda^2 E \otimes \mathcal{L}) \oplus (\Lambda^4 E \otimes \mathcal{L}^2) \oplus \dots$$

and equipped with an algebra structure via the Clifford multiplication. If the dimension  $n$  of fibers of  $X \rightarrow S$  is odd, then  $\mathcal{C}_0$  is a sheaf of Azumaya algebras on the open subset of  $S$  corresponding to nondegenerate quadrics (which of course may be empty). On the other hand, if  $n$  is even then  $\mathcal{O}_S \oplus \Lambda^n E \otimes \mathcal{L}^{n/2}$  is a central subalgebra in  $\mathcal{C}_0$ , so the latter gives a sheaf  $\tilde{\mathcal{C}}_0$  of algebras on the twofold covering

$$\tilde{S} := \text{Spec}_S(\mathcal{O}_S \oplus \Lambda^n E \otimes \mathcal{L}^{n/2}) \quad (2.14)$$

of  $S$ , and  $\tilde{\mathcal{C}}_0$  is a sheaf of Azumaya algebras on the preimage of the open subset of  $S$  corresponding to nondegenerate quadrics.

**2.4. Base change.** A triangulated category  $\mathcal{T}$  is  $S$ -linear if it is equipped with a module structure over the tensor triangulated category  $\mathbf{D}^b(\text{coh}(S))$ . In particular, if  $X$  is a scheme over  $S$  and  $f : X \rightarrow S$  is the structure morphism then a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle \tag{2.15}$$

is  $S$ -linear if each of subcategories  $\mathcal{A}_k$  is closed under tensoring with an object of  $\mathbf{D}^b(\text{coh}(S))$ , i.e. for  $A \in \mathcal{A}_k$  and  $F \in \mathbf{D}^b(\text{coh}(S))$  one has  $A \otimes_{\mathbb{L}} Lf^*(F) \in \mathcal{A}_k$ .

The semiorthogonal decompositions of Theorems 2.4, 2.5 and 2.7 are  $S$ -linear, and the blowup formula of Theorem 2.6 is  $X$ -linear. The advantage of linear semiorthogonal decompositions lies in the fact that they obey a base change result. For a base change  $T \rightarrow S$  denote by  $\pi : X \times_S T \rightarrow X$  the induced projection.

**Theorem 2.8** ([37]). *If  $X$  is an algebraic variety over  $S$  and (2.15) is an  $S$ -linear semiorthogonal decomposition then for a change of base morphism  $T \rightarrow S$  there is, under a certain technical condition, a  $T$ -linear semiorthogonal decomposition*

$$\mathbf{D}^b(\text{coh}(X \times_S T)) = \langle \mathcal{A}_{1T}, \dots, \mathcal{A}_{mT} \rangle$$

such that  $\pi^*(A) \subset \mathcal{A}_{iT}$  for any  $A \in \mathcal{A}_i$  and  $\pi_*(A') \subset \mathcal{A}_i$  for any  $A' \in \mathcal{A}_{iT}$  which has proper support over  $X$ .

**2.5. Important questions.** There are several questions which might be crucial for further investigations.

**Question 2.9.** *Find a good condition for an exceptional collection to be full.*

One might hope that if the collection generates the Grothendieck group (or the Hochschild homology) of the category then it is full. However, recent examples of quasiphantom and phantom categories (see section 6.2) show that this is not the case. Still we may hope that in the categories generated by exceptional collections there are no phantoms. In other words one could hope that the following is true.

**Conjecture 2.10.** *Let  $\mathcal{T} = \langle E_1, \dots, E_n \rangle$  be a triangulated category generated by an exceptional collection. Then any exceptional collection of length  $n$  in  $\mathcal{T}$  is full.*

If there is an action of a group  $G$  on an algebraic variety  $X$ , one can consider the equivariant derived category  $\mathbf{D}^b(\text{coh}^G(X))$  along with the usual derived category. In many interesting cases (flag varieties, toric varieties, GIT quotients) it is quite easy to construct a full exceptional collection in the equivariant category. It would be extremely useful to find a way to transform it into a full exceptional collection in the usual category. In some sense the results of [41] can be considered as an example of such an approach.

Another very important question is to find possible restrictions for existence of semiorthogonal decompositions. Up to now there are only several cases when we can *prove* indecomposability of a category. The first is the derived category of a curve of positive genus. The proof (see e.g. [44]) is based on special properties of categories of homological dimension 1. Another is the derived category of a Calabi–Yau variety (smooth connected variety with trivial canonical class). Its indecomposability is proved by a surprisingly simple argument due to Bridgeland [8]. This was further generalized in [21] to varieties with globally generated canonical class. On the other hand, the original argument of Bridgeland generalizes to any connected Calabi–Yau category (i.e. with the Serre functor isomorphic to a shift and Hochschild cohomology in degree zero isomorphic to  $k$ ).

### 3. Homological projective duality

The starting point of a homological projective duality (HP duality for short) is a smooth projective variety  $X$  with a morphism into a projective space and a semiorthogonal decomposition of  $\mathbf{D}^b(\text{coh}(X))$  of a very special type.

**3.1. Lefschetz decompositions.** Let  $X$  be an algebraic variety and  $\mathcal{L}$  a line bundle on  $X$ .

**Definition 3.1.** A right Lefschetz decomposition of  $\mathbf{D}^b(\text{coh}(X))$  with respect to  $\mathcal{L}$  is a semiorthogonal decomposition of form

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{L}, \dots, \mathcal{A}_{m-1} \otimes \mathcal{L}^{m-1} \rangle \tag{3.1}$$

with  $0 \subset \mathcal{A}_{m-1} \subset \dots \subset \mathcal{A}_1 \subset \mathcal{A}_0$ . In other words, each component of the decomposition is a subcategory of the previous component twisted by  $\mathcal{L}$ .

Analogously, a left Lefschetz decomposition of  $\mathbf{D}^b(\text{coh}(X))$  with respect to  $\mathcal{L}$  is a semiorthogonal decomposition of form

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{B}_{m-1} \otimes \mathcal{L}^{1-m}, \dots, \mathcal{B}_1 \otimes \mathcal{L}^{-1}, \mathcal{B}_0 \rangle \tag{3.2}$$

with  $0 \subset \mathcal{B}_{m-1} \subset \dots \subset \mathcal{B}_1 \subset \mathcal{B}_0$ .

The subcategories  $\mathcal{A}_i$  (resp.  $\mathcal{B}_i$ ) forming a Lefschetz decomposition will be called blocks, the largest will be called the first block. Usually we will consider right Lefschetz decompositions. So, we will call them simply Lefschetz decompositions.

Beilinson’s collection on  $\mathbb{P}^n$  is an example of a Lefschetz decomposition with  $\mathcal{A}_0 = \mathcal{A}_1 = \dots = \mathcal{A}_n = \langle \mathcal{O}_{\mathbb{P}^n} \rangle$ . Kapranov’s collection on the Grassmannian  $\text{Gr}(k, n)$  also has a Lefschetz structure with the category  $\mathcal{A}_i$  generated by  $\Sigma^\alpha \mathcal{U}^\vee$  for  $\alpha \subset R(k-1, n-k-i)$ .

Note that in Definition 3.1 one can replace the twist by a line bundle with any other autoequivalence of  $\mathbf{D}^b(\text{coh}(X))$  and get the notion of a Lefschetz decomposition with respect to an autoequivalence. This may be especially useful when dealing with arbitrary triangulated categories.

It is also useful to know that for a given line bundle  $\mathcal{L}$  a Lefschetz decomposition is completely determined by its first block. Moreover, an admissible subcategory extends to a right Lefschetz decomposition if and only if it extends to a left Lefschetz decomposition. The simplest example of an admissible subcategory which does not extend to a Lefschetz decomposition is the subcategory  $\langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2) \rangle \subset \mathbf{D}^b(\text{coh}(\mathbb{P}^2))$ .

**Question 3.2.** Find a good sufficient condition for a Lefschetz extendability of an admissible subcategory  $\mathcal{A}_0 \subset \mathbf{D}^b(\text{coh}(X))$ .

One can define a partial ordering on the set of all Lefschetz decompositions of  $\mathbf{D}^b(\text{coh}(X))$  by inclusions of their first blocks. As we will see soon, the most interesting and strong results are obtained by using minimal Lefschetz decompositions.

**3.2. Hyperplane sections.** Let  $X$  be a smooth projective variety with a morphism into a projective space  $f : X \rightarrow \mathbb{P}(V)$  (not necessarily an embedding). Put  $\mathcal{O}_X(1) := f^* \mathcal{O}_{\mathbb{P}(V)}(1)$  and assume that a right Lefschetz decomposition with respect to  $\mathcal{O}_X(1)$

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{m-1}(m-1) \rangle \tag{3.3}$$

is given (we abbreviate  $\mathcal{A}_i(i) := \mathcal{A}_i \otimes \mathcal{O}_X(i)$ ). Consider the dual projective space  $\mathbb{P}(V^\vee)$ . By the base change (Theorem 2.8) the product  $X \times \mathbb{P}(V^\vee)$  inherits a  $\mathbb{P}(V^\vee)$ -linear semiorthogonal decomposition

$$\mathbf{D}^b(\mathrm{coh}(X \times \mathbb{P}(V^\vee))) = \langle \mathcal{A}_{0\mathbb{P}(V^\vee)}, \mathcal{A}_1(1)_{\mathbb{P}(V^\vee)}, \dots, \mathcal{A}_{m-1}(m-1)_{\mathbb{P}(V^\vee)} \rangle$$

Consider the universal hyperplane section of  $X$ ,  $\mathcal{X} := X \times_{\mathbb{P}(V)} Q \subset X \times \mathbb{P}(V^\vee)$ , where  $Q \subset \mathbb{P}(V) \times \mathbb{P}(V^\vee)$  is the incidence quadric and denote by  $\alpha : \mathcal{X} \rightarrow X \times \mathbb{P}(V^\vee)$  the natural embedding.

**Lemma 3.3.** *The functor  $L\alpha^* : \mathbf{D}^b(\mathrm{coh}(X \times \mathbb{P}(V^\vee))) \rightarrow \mathbf{D}^b(\mathrm{coh}(\mathcal{X}))$  is fully faithful on each of the subcategories  $\mathcal{A}_1(1)_{\mathbb{P}(V^\vee)}, \dots, \mathcal{A}_{m-1}(m-1)_{\mathbb{P}(V^\vee)}$  and induces a  $\mathbb{P}(V^\vee)$ -linear semiorthogonal decomposition*

$$\mathbf{D}^b(\mathrm{coh}(\mathcal{X})) = \langle \mathcal{C}, \mathcal{A}_1(1)_{\mathbb{P}(V^\vee)}, \dots, \mathcal{A}_{m-1}(m-1)_{\mathbb{P}(V^\vee)} \rangle. \tag{3.4}$$

The first component  $\mathcal{C}$  of this decomposition is called the HP dual category of  $X$ . It is a very interesting category, especially if it can be identified with the derived category of some algebraic variety  $Y$ . In this case this variety is called the HP dual variety of  $X$ .

**Definition 3.4.** An algebraic variety  $Y$  equipped with a morphism  $g : Y \rightarrow \mathbb{P}(V^\vee)$  is called homologically projectively dual to  $f : X \rightarrow \mathbb{P}(V)$  with respect to a given Lefschetz decomposition (3.3) if there is given an object  $\mathcal{E} \in \mathbf{D}^b(\mathrm{coh}(Q(X, Y)))$  such that the Fourier–Mukai functor  $\Phi_{\mathcal{E}} : \mathbf{D}^b(\mathrm{coh}(Y)) \rightarrow \mathbf{D}^b(\mathrm{coh}(\mathcal{X}))$  is an equivalence onto the HP dual subcategory  $\mathcal{C} \subset \mathbf{D}^b(\mathrm{coh}(\mathcal{X}))$  of (3.4).

Here  $Q(X, Y) = (X \times Y) \times_{\mathbb{P}(V) \times \mathbb{P}(V^\vee)} Q = \mathcal{X} \times_{\mathbb{P}(V^\vee)} Y$ . If a homological projective duality between varieties  $X$  and  $Y$  is established then there is an interesting relation between derived categories of their linear sections.

**3.3. Homologically projectively duality statement.** For each linear subspace  $L \subset V^\vee$  denote its orthogonal complement in  $V$  by  $L^\perp := \mathrm{Ker}(V \rightarrow L^\vee)$ . Further denote

$$X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp), \quad Y_L := Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L). \tag{3.5}$$

Varieties defined in this way are called mutually orthogonal linear sections of  $X$  and  $Y$ . We will say that  $X_L$  and  $Y_L$  have expected dimensions if

$$\dim X_L = \dim X - r \quad \text{and} \quad \dim Y_L = \dim Y - (N - r),$$

where  $N = \dim V$  and  $r = \dim L$  (so that  $N - r = \dim L^\perp$ ).

**Theorem 3.5** (Homological projective duality, [29]). *Let  $(Y, g)$  be an HP dual variety for  $(X, f)$  with respect to (3.3). Then*

- (1)  *$Y$  is smooth and  $\mathbf{D}^b(\mathrm{coh}(Y))$  has an admissible subcategory  $\mathcal{B}_0$  equivalent to  $\mathcal{A}_0$  and extending to a left Lefschetz decomposition*

$$\mathbf{D}^b(\mathrm{coh}(Y)) = \langle \mathcal{B}_{n-1}(1-n), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle, \quad \mathcal{B}_{n-1} \subset \dots \subset \mathcal{B}_1 \subset \mathcal{B}_0. \tag{3.6}$$

- (2)  *$(X, f)$  is HP dual to  $(Y, g)$  with respect to (3.6).*

- (3) The set of critical values of  $g$  is the classical projective dual of  $X$ .
- (4) For any subspace  $L \subset V^\vee$  if  $X_L$  and  $Y_L$  have expected dimensions then there are semiorthogonal decompositions

$$\mathbf{D}^b(\text{coh}(X_L)) = \langle \mathcal{C}_L, \mathcal{A}_r(r), \dots, \mathcal{A}_{m-1}(m-1) \rangle, \tag{3.7}$$

$$\mathbf{D}^b(\text{coh}(Y_L)) = \langle \mathcal{B}_{n-1}(1-n), \dots, \mathcal{B}_{N-r}(r-N), \mathcal{C}_L \rangle \tag{3.8}$$

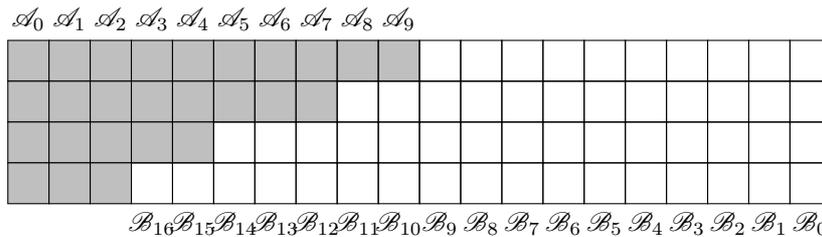
with the same triangulated category  $\mathcal{C}_L$  appearing in the RHS.

The decomposition (3.6) of  $\mathbf{D}^b(\text{coh}(Y))$  will be referred to as the HP dual Lefschetz decomposition. The common component  $\mathcal{C}_L$  of decompositions (3.7) and (3.8) will be referred to as the nontrivial part of the derived categories of  $X_L$  and  $Y_L$ , while the subcategories  $\mathcal{A}_i(i)$  and  $\mathcal{B}_j(-j)$  (one checks that the pullback functors for the embeddings  $X_L \rightarrow X$  and  $Y_L \rightarrow Y$  are fully faithful on the subcategories  $\mathcal{A}_i$  and  $\mathcal{B}_j$  for  $i \geq r$  and  $j \geq N - r$ ) are considered as trivial (in the sense that they come from the ambient varieties).

The first two statements of this Theorem show that the relation we are dealing with is indeed a duality, the third statement shows the relation to the classical projective duality (and so justifies the word “projective” in the name), and the last statement is the real result. We will soon see how powerful it is.

Note also that in the statement of the Theorem the linear sections  $X_L$  and  $Y_L$  need not be smooth. In fact, one can show that for HP dual varieties  $X$  and  $Y$  a section  $X_L$  is smooth if and only if its orthogonal section  $Y_L$  is smooth, but no matter whether this is the case or not, the decompositions (3.7) and (3.8) hold true.

Now let us say some words about the relations of the Lefschetz decompositions (3.3) and (3.6) for HP dual varieties. As it was already mentioned, the largest components of those are just equivalent  $\mathcal{B}_0 \cong \mathcal{A}_0$ . Further, the component  $\mathcal{B}_i$  is very close to the orthogonal complement of  $\mathcal{A}_{N-1-i}$  in  $\mathcal{A}_0$ . More precisely, these two categories have semiorthogonal decompositions with the same components but with in general different gluing functors. This can be visualized by a picture.



The gray part of the picture corresponds to the initial Lefschetz decomposition, the columns correspond to its blocks, while the white part corresponds to the dual decomposition, the complementary columns correspond to the complementary subcategories of the dual Lefschetz decomposition. The number of rows is equal to the number of different components in the initial (and the dual) Lefschetz decomposition. In this example picture  $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 \neq \mathcal{A}_3 = \mathcal{A}_4 \neq \mathcal{A}_5 = \mathcal{A}_6 = \mathcal{A}_7 \neq \mathcal{A}_8 = \mathcal{A}_9$ , and so one can say that the rows correspond to the “primitive parts”  $(\mathcal{A}_3)^\perp_{\mathcal{A}_0}$ ,  $(\mathcal{A}_5)^\perp_{\mathcal{A}_3}$ ,  $(\mathcal{A}_8)^\perp_{\mathcal{A}_5}$ , and  $\mathcal{A}_8$  of all the categories in the picture, the length of the initial decomposition is  $m = 10$ , the length of the dual decomposition is  $n = 17$ , while the dimension of the ambient space is  $N = 20$ .

Note that  $\mathcal{B}_i = 0$  if and only if  $\mathcal{A}_{N-1-i} = \mathcal{A}_0$ , so the number  $n$  of components in (3.6) equals  $N$  minus the number of components in (3.3) equal to  $\mathcal{A}_0$ .

In fact, the best (in many aspects) situation is when in the original Lefschetz decomposition (3.3) all components coincide  $\mathcal{A}_0 = \mathcal{A}_1 = \dots = \mathcal{A}_{m-1}$  (such Lefschetz decompositions are called rectangular). Then the HP dual Lefschetz decomposition is also rectangular, has the same components  $\mathcal{B}_0 = \mathcal{B}_1 = \dots = \mathcal{B}_{n-1} \cong \mathcal{A}_0$  and

$$n = N - m$$

(in particular in a picture analogous to the above the gray and the white parts are rectangles, which explains the name “rectangular”). Moreover, in this case for any  $0 < r < N$  one has either  $r \geq m$  or  $N - r \geq n$ , hence in decompositions (3.7) and (3.8) either the first or the second category has only one component  $\mathcal{C}_L$  and nothing else. Then the other decomposition shows that the nontrivial component of the derived category of a linear section is equivalent to the derived category of the orthogonal linear section of the dual variety.

**3.4. HP duality and noncommutative varieties.** In general, the homologically projectively dual category  $\mathcal{C} \subset \mathbf{D}^b(\text{coh}(\mathcal{X}))$  defined by (3.4) need not be equivalent to  $\mathbf{D}^b(\text{coh}(Y))$  for an algebraic variety  $Y$ . In fact, only a few such cases are known — the linear duality, the duality for quadrics, the duality for Grassmannians  $\text{Gr}(2, 4)$  and  $\text{Gr}(2, 5)$ , and the spinor variety  $\mathbb{S}_5$  (see section 5).

One can get many additional interesting examples by allowing  $Y$  to be a noncommutative variety. Here a noncommutative variety can be understood in different ways. If one uses the most general sense — as a semiorthogonal component of the derived category of an algebraic variety — then tautologically the HP dual category  $\mathcal{C}$  itself will provide a noncommutative HP dual variety. In fact, one can develop a theory of HP duality for noncommutative varieties in this sense and prove the same results (see [30]). However, in this most general form the semiorthogonal decompositions provided by the HP duality Theorem will not have an apparent geometric interpretation.

In fact, an interesting geometry arises in HP duality if the dual variety  $Y$  is close to a commutative variety. For example, it often happens that there is a (commutative) algebraic variety  $Y_0$  with a map  $g_0 : Y_0 \rightarrow \mathbb{P}(V^\vee)$ , a sheaf of finite  $\mathcal{O}_{Y_0}$ -algebras  $\mathcal{R}$  on  $Y_0$  whose bounded derived category  $\mathbf{D}^b(\text{coh}(Y_0, \mathcal{R}))$  of coherent  $\mathcal{R}$ -modules on  $Y_0$  is equivalent to the HP dual category  $\mathcal{C}$  of  $X$  and such that the equivalence  $\mathcal{C} \cong \mathbf{D}^b(\text{coh}(Y_0, \mathcal{R}))$  is given by an appropriate object  $\mathcal{E} \in \mathbf{D}^b(\text{coh}(Q(X, Y_0), \mathcal{R}))$ . Of course, one can easily allow here the variety  $X$  also to be noncommutative in the same sense. It is easy to modify all the definitions accordingly.

In section 5 we discuss examples showing that this generalization is meaningful. Among such examples are the Veronese–Clifford duality, the Grassmannian–Pfaffian duality, and their generalizations.

In fact, in some of these examples, the HP duality Theorem 3.5 still gives semiorthogonal decompositions for usual commutative varieties (even though the dual variety is noncommutative). Indeed, the sheaf of algebras  $\mathcal{R}$  on  $Y_0$  is frequently isomorphic to a matrix algebra on an open subset of  $Y_0$ , typically, on its smooth locus — in fact, in these cases the noncommutative variety  $(Y_0, \mathcal{R})$  can be thought of as a categorical resolution of singularities of  $Y$ . In this situation, taking a subspace  $L \subset V^\vee$  such that  $Y_{0L}$  is contained in that open subset, one gets  $\mathbf{D}^b(\text{coh}(Y_L)) = \mathbf{D}^b(\text{coh}(Y_{0L}, \mathcal{R})) \cong \mathbf{D}^b(\text{coh}(Y_{0L}))$ .

#### 4. Categorical resolutions of singularities

As it was explained above (and we will see in some of the examples below) in many cases the HP dual variety looks as a noncommutative (or categorical) resolution of singularities of a singular variety. So, a good notion of a categorical resolution is necessary for the theory.

**4.1. The definition.** If  $\pi : \tilde{Y} \rightarrow Y$  is a resolution of singularities (in the usual sense), we have an adjoint pair of triangulated functors

$$R\pi_* : \mathbf{D}^b(\mathrm{coh}(\tilde{Y})) \rightarrow \mathbf{D}^b(\mathrm{coh}(Y))$$

and

$$L\pi^* : \mathbf{D}^{\mathrm{perf}}(Y) \rightarrow \mathbf{D}^b(\mathrm{coh}(\tilde{Y}))$$

(here  $\mathbf{D}^{\mathrm{perf}}(Y)$  stands for the category of perfect complexes on  $Y$ ). We axiomatize this situation in the following

**Definition 4.1** (cf. [33, 39]). A categorical resolution of singularities of a scheme  $Y$  is a smooth triangulated category  $\mathcal{T}$  and an adjoint pair of triangulated functors

$$\pi_* : \mathcal{T} \rightarrow \mathbf{D}^b(\mathrm{coh}(Y))$$

and

$$\pi^* : \mathbf{D}^{\mathrm{perf}}(Y) \rightarrow \mathcal{T}$$

such that  $\pi_* \circ \pi^* \cong \mathrm{id}_{\mathbf{D}^{\mathrm{perf}}(Y)}$ . In particular, the functor  $\pi^*$  is fully faithful.

We will not discuss the notion of smoothness for a triangulated category. In fact, for our purposes it is always enough to assume that  $\mathcal{T}$  is an admissible  $Y$ -linear subcategory of  $\mathbf{D}^b(\mathrm{coh}(\tilde{Y}))$  for a geometric resolution  $\tilde{Y} \rightarrow Y$ .

Let  $(\mathcal{T}, \pi_*, \pi^*)$  and  $(\mathcal{T}', \pi'_*, \pi'^*)$  be two categorical resolutions of  $Y$ . We will say that  $\mathcal{T}$  dominates  $\mathcal{T}'$  if there is a fully faithful functor  $\epsilon : \mathcal{T}' \rightarrow \mathcal{T}$  such that  $\pi'_* = \pi_* \circ \epsilon$ . Clearly, this is compatible with the usual dominance relation between geometric resolutions — if a resolution  $\pi : \tilde{Y} \rightarrow Y$  factors as  $\tilde{Y} \xrightarrow{f} \tilde{Y}' \xrightarrow{\pi'} Y$  then the pullback functor  $\epsilon := Lf^* : \mathbf{D}^b(\mathrm{coh}(\tilde{Y}')) \rightarrow \mathbf{D}^b(\mathrm{coh}(\tilde{Y}))$  is fully faithful and

$$R\pi_* \circ Lf^* = R\pi'_* \circ Rf_* \circ Lf^* \cong R\pi'_*.$$

Categorical resolutions have two advantages in comparison with geometric ones. First, if  $Y$  has irrational singularities the pullback functor for a geometric resolution is never fully faithful and so its derived category is not a categorical resolution in sense of Definition 4.1. However, it was shown in [39] that any separated scheme of finite type (even nonreduced) over a field of zero characteristic admits a categorical resolution.

The second advantage is that the dominance order for categorical resolutions is more flexible. For example, in many examples one can find a categorical resolution which is much smaller than any geometric resolution. There are strong indications that the Minimal Model Program on the categorical level may be much simpler than the classical one. In particular, we expect the following.

**Conjecture 4.2** (cf. [7]). *For any quasiprojective scheme  $Y$  there exists a categorical resolution which is minimal with respect to the dominance order.*

**4.2. Examples of categorical resolutions.** As an evidence for the conjecture we will construct categorical resolutions which are presumably minimal.

**Theorem 4.3** ([33]). *Let  $f : \tilde{Y} \rightarrow Y$  be a resolution of singularities and let  $E$  be the exceptional divisor with  $i : E \rightarrow \tilde{Y}$  being the embedding. Assume that the derived category  $\mathbf{D}^b(\text{coh}(E))$  has a left Lefschetz decomposition with respect to the conormal bundle  $\mathcal{O}_E(-E)$ :*

$$\mathbf{D}^b(\text{coh}(E)) = \langle \mathcal{C}_{m-1}((m-1)E), \dots, \mathcal{C}_1(E), \mathcal{C}_0 \rangle, \tag{4.1}$$

which is  $Y$ -linear and has  $Li^*(Lf^*(\mathbf{D}^{\text{perf}}(Y))) \subset \mathcal{C}_0$ . Then the functor  $Ri_*$  is fully faithful on subcategories  $\mathcal{C}_k \subset \mathbf{D}^b(\text{coh}(E))$  for  $k > 0$ , the subcategory

$$\tilde{\mathcal{C}} := \{F \in \mathbf{D}^b(\text{coh}(\tilde{Y})) \mid Li^*(F) \in \mathcal{C}_0\} \tag{4.2}$$

is admissible in  $\mathbf{D}^b(\text{coh}(\tilde{Y}))$ , and there is a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(\tilde{Y})) = \langle Ri_*(\mathcal{C}_{m-1}((m-1)E)), \dots, Ri_*(\mathcal{C}_1(E)), \tilde{\mathcal{C}} \rangle. \tag{4.3}$$

Moreover, the functor  $Lf^* : \mathbf{D}^{\text{perf}}(Y) \rightarrow \mathbf{D}^b(\text{coh}(\tilde{Y}))$  factors as a composition of a fully faithful functor  $\pi^* : \mathbf{D}^{\text{perf}}(Y) \rightarrow \tilde{\mathcal{C}}$  with the embedding  $\gamma : \tilde{\mathcal{C}} \rightarrow \mathbf{D}^b(\text{coh}(\tilde{Y}))$ , and the functors  $\pi_* := Rf_* \circ \gamma$  and  $\pi^*$  give  $\tilde{\mathcal{C}}$  a structure of a categorical resolution of singularities of  $Y$ .

If  $\mathcal{C}'_0 \subset \mathcal{C}_0 \subset \mathbf{D}^b(\text{coh}(E))$  are two admissible Lefschetz extendable subcategories (with respect to the conormal bundle) then clearly by (4.2) the categorical resolution  $\tilde{\mathcal{C}}'$  constructed from  $\mathcal{C}'_0$  is a subcategory in the categorical resolution  $\tilde{\mathcal{C}}$  constructed from  $\mathcal{C}_0$ . Moreover, if  $\epsilon : \tilde{\mathcal{C}}' \rightarrow \tilde{\mathcal{C}}$  is the embedding functor then  $\pi'_* = \pi_* \circ \epsilon$ , so  $\tilde{\mathcal{C}}$  dominates  $\tilde{\mathcal{C}}'$ . This shows that minimal categorical resolutions are related to minimal Lefschetz decompositions.

As an example of the application of the above Theorem consider the cone  $Y$  over a smooth projective variety  $X \subset \mathbb{P}(V)$ . Then  $\tilde{Y} = \text{Tot}_X(\mathcal{O}_X(-1))$ , the total space of the line bundle  $\mathcal{O}_X(-1) = \mathcal{O}_{\mathbb{P}(V)}(-1)|_X$ , is a geometric resolution of  $Y$ . The exceptional divisor of the natural morphism  $f : \tilde{Y} \rightarrow Y$  then identifies with the zero section of the total space,  $E = X$ , and the conormal bundle identifies with  $\mathcal{O}_X(1)$ . So, a left Lefschetz decomposition of  $\mathbf{D}^b(\text{coh}(X))$  with respect to  $\mathcal{O}_X(1)$  gives a categorical resolution of the cone  $Y$  over  $X$ .

**Example 4.4.** Take  $X = \mathbb{P}^3$  with the double Veronese embedding  $f : \mathbb{P}^3 \rightarrow \mathbb{P}^9$ , so that  $f^*\mathcal{O}_{\mathbb{P}^9}(1) = \mathcal{O}_{\mathbb{P}^3}(2)$ , and a left Lefschetz decomposition

$$\mathbf{D}^b(\text{coh}(\mathbb{P}^3)) = \langle \mathcal{C}_1(-2), \mathcal{C}_0 \rangle \quad \text{with } \mathcal{C}_0 = \mathcal{C}_1 = \langle \mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{O}_{\mathbb{P}^3} \rangle.$$

Then the category  $\tilde{\mathcal{C}} := \{F \in \mathbf{D}^b(\text{coh}(\text{Tot}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(-2)))) \mid Li^*F \in \langle \mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{O}_{\mathbb{P}^3} \rangle\}$  is a categorical resolution of the Veronese cone, which is significantly smaller than the usual geometric resolution. It is expected to be minimal.

**4.3. Crepancy of categorical resolutions.** Crepancy is an important property of a resolution which in the geometric situation ensures its minimality. A resolution  $f : \tilde{Y} \rightarrow Y$  is crepant if the relative canonical class  $K_{\tilde{Y}/Y}$  is trivial. There is an analogue of crepancy for categorical resolutions. In fact, there are two such analogues.

**Definition 4.5** ([33]). A categorical resolution  $(\mathcal{T}, \pi_*, \pi^*)$  of a scheme  $Y$  is weakly crepant if the functor  $\pi^* : \mathbf{D}^{\text{perf}}(Y) \rightarrow \mathcal{T}$  is simultaneously left and right adjoint to the functor  $\pi_* : \mathcal{T} \rightarrow \mathbf{D}^b(\text{coh}(Y))$ .

By Grothendieck duality, the right adjoint of the derived pushforward functor  $Rf_* : \mathbf{D}^b(\text{coh}(\tilde{Y})) \rightarrow \mathbf{D}^b(\text{coh}(Y))$  is given by  $f^!(F) = Lf^*(F) \otimes_{\mathcal{O}_{\tilde{Y}}} (K_{\tilde{Y}/Y})$ , so for a geometric resolution crepancy and weak crepancy are equivalent.

**Definition 4.6** ([33]). A categorical resolution  $(\mathcal{T}, \pi_*, \pi^*)$  of a scheme  $Y$  is strongly crepant if the relative Serre functor of  $\mathcal{T}$  over  $\mathbf{D}^b(\text{coh}(Y))$  is isomorphic to the identity.

Again, Grothendieck duality implies that for a geometric resolution crepancy and strong crepancy are equivalent. Moreover, it is not so difficult to show that strong crepancy of a categorical resolution implies its weak crepancy, but the converse is not true in general. To see this one can analyze the weak and strong crepancy of categorical resolutions provided by Theorem 4.3.

**Proposition 4.7.** *In the setup of Theorem 4.3 assume that the scheme  $Y$  is Gorenstein and  $K_{\tilde{Y}/Y} = (m - 1)E$ , where  $m$  is the length of the left Lefschetz decomposition (4.1). The corresponding categorical resolution  $\tilde{\mathcal{C}}$  of  $Y$  is weakly crepant if and only if*

$$Li^*(Lf^*(\mathbf{D}^{\text{perf}}(Y))) \subset \mathcal{C}_{m-1}. \tag{4.4}$$

Furthermore,  $\tilde{\mathcal{C}}$  is strongly crepant if and only if (4.1) is rectangular, i.e.

$$\mathcal{C}_{m-1} = \dots = \mathcal{C}_1 = \mathcal{C}_0. \tag{4.5}$$

So, starting from a nonrectangular Lefschetz decomposition it is easy to produce an example of a weakly crepant categorical resolution which is not strongly crepant.

**Example 4.8.** Take  $X = Q^3 \subset \mathbb{P}^4$  and let  $Y$  be the cone over  $X$  (i.e. a 4-dimensional quadratic cone). Then the left Lefschetz collection

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{C}_2(-2), \mathcal{C}_1(-1), \mathcal{C}_0 \rangle \quad \text{with } \mathcal{C}_0 = \langle \mathcal{S}, \mathcal{O}_X \rangle, \mathcal{C}_1 = \mathcal{C}_2 = \langle \mathcal{O}_X \rangle$$

( $\mathcal{S}$  is the spinor bundle) gives a weakly crepant categorical resolution  $\tilde{\mathcal{C}}$  of  $Y$  which is not strongly crepant. In fact, if  $q : \text{Tot}_X(\mathcal{O}_X(-1)) \rightarrow X$  is the canonical projection, the vector bundle  $q^*\mathcal{S}$  is a spherical object in  $\tilde{\mathcal{C}}$  and the relative Serre functor is isomorphic to the corresponding spherical twist.

**4.4. Further questions.** Of course, the central question is Conjecture 4.2. Theorem 4.3 shows that it is closely related to the question of existence of minimal Lefschetz decompositions.

Another interesting question is to find new methods of construction of minimal categorical resolutions. An interesting development in this direction is the work [1] in which a notion of a wonderful resolution of singularities (an analogue of wonderful compactifications) is introduced and it is shown that a wonderful resolution gives rise to a weakly crepant categorical resolution. This can be viewed as an advance on the first part of Proposition 4.7. It would be very interesting to find a generalization of the second part of this Proposition in the context of wonderful resolutions.

Another aspect is to find explicit constructions of minimal resolutions for interesting varieties, such as Pfaffian varieties for example. Some of these arise naturally in the context of HP duality as we will see later.



**Theorem 5.2.**

(1) If  $X = Q^{2m} \subset \mathbb{P}^{2m+1}$  with Lefschetz decomposition given by

$$\mathcal{A}_0 = \mathcal{A}_1 = \langle S_X, \mathcal{O}_X \rangle, \quad \mathcal{A}_2 = \mathcal{A}_3 = \dots = \mathcal{A}_{2m-1} = \langle \mathcal{O}_X \rangle \quad (5.1)$$

then the HP dual variety is the dual quadric  $Y = Q^\vee \subset \check{\mathbb{P}}^{2m+1}$  with the same Lefschetz decomposition.

$$\begin{array}{c} \mathcal{O}_X \\ S_X \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \cdots \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \begin{array}{c} S_Y \\ \mathcal{O}_Y \end{array}$$

(2) If  $X = Q^{2m-1} \subset \mathbb{P}^{2m}$  with Lefschetz decomposition given by

$$\mathcal{A}_0 = \langle S_X, \mathcal{O}_X \rangle, \quad \mathcal{A}_1 = \mathcal{A}_2 = \dots = \mathcal{A}_{2m-2} = \langle \mathcal{O}_X \rangle \quad (5.2)$$

then the HP dual variety is the double covering  $Y \rightarrow \check{\mathbb{P}}^{2m}$  ramified in the dual quadric  $Q^\vee \subset \check{\mathbb{P}}^{2m}$  with Lefschetz decomposition (5.1)

$$\begin{array}{c} \mathcal{O}_X \\ S_X \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \cdots \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \begin{array}{c} S_Y \\ \mathcal{O}_Y \end{array}$$

(3) If  $X = Q^{2m-1} \rightarrow \mathbb{P}^{2m-1}$  is the double covering ramified in a quadric  $\bar{Q} \subset \mathbb{P}^{2m-1}$  with Lefschetz decomposition (5.2) then the HP dual variety is the double covering  $Y \rightarrow \check{\mathbb{P}}^{2m-1}$  ramified in the dual quadric  $\bar{Q}^\vee \subset \check{\mathbb{P}}^{2m-1}$  with the same Lefschetz decomposition.

$$\begin{array}{c} \mathcal{O}_X \\ S_X \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \cdots \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \begin{array}{c} S_Y \\ \mathcal{O}_Y \end{array}$$

**5.3. Veronese–Clifford duality.** Let  $W$  be a vector space of dimension  $n$  and  $V = S^2W$  its symmetric square. We take  $X = \mathbb{P}(W)$  and consider its double Veronese embedding  $f : \mathbb{P}(W) \rightarrow \mathbb{P}(V)$ . Then  $f^* \mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{O}_{\mathbb{P}(W)}(2)$ . Beilinson’s collection (2.6) on  $\mathbb{P}(W)$  can be considered as a Lefschetz decomposition (with respect to  $\mathcal{O}_{\mathbb{P}(W)}(2)$ ) of  $\mathbf{D}^b(\text{coh}(\mathbb{P}(W)))$  with  $\lfloor n/2 \rfloor$  blocks equal to

$$\mathcal{A}_0 = \mathcal{A}_1 = \dots = \mathcal{A}_{\lfloor n/2 \rfloor - 1} := \langle \mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)}(1) \rangle,$$

and if  $n$  is odd one more block

$$\mathcal{A}_{\lfloor n/2 \rfloor} := \langle \mathcal{O}_{\mathbb{P}(W)} \rangle.$$

The universal hyperplane section  $\mathcal{X}$  of  $X$  is nothing but the universal quadric in  $\mathbb{P}(W)$  over the space  $\mathbb{P}(V^\vee) = \mathbb{P}(S^2W^\vee)$  of all quadrics. Then the quadratic fibration formula of Theorem 2.7 gives an equivalence of the HP dual category  $\mathcal{C}$  with the derived category  $\mathbf{D}^b(\text{coh}(\mathbb{P}(V^\vee), \mathcal{C}_0))$  of coherent sheaves of modules over the even part of the universal Clifford algebra

$$\mathcal{C}_0 = \mathcal{O}_{\mathbb{P}(S^2W^\vee)} \oplus \Lambda^2 W \otimes \mathcal{O}_{\mathbb{P}(S^2W^\vee)}(-1) \oplus \dots,$$

on the space  $\mathbb{P}(S^2W^\vee)$  of quadrics. We will consider the pair  $(\mathbb{P}(S^2W^\vee), \mathcal{C}_0)$  as a non-commutative variety and call it the Clifford space.

**Theorem 5.3** (Veronese–Clifford duality, [31]). *The homological projective dual of the projective space  $X = \mathbb{P}(W)$  in the double Veronese embedding  $\mathbb{P}(W) \rightarrow \mathbb{P}(S^2W)$  is the Clifford space  $Y = (\mathbb{P}(S^2W^\vee), \mathcal{C}_0)$ .*

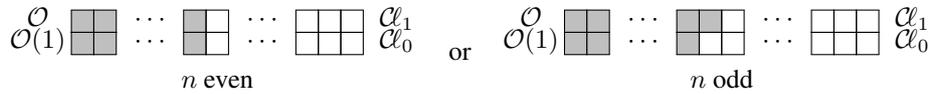
The HP dual Lefschetz decomposition of the Clifford space is given by the full exceptional collection

$$\mathbf{D}^b(\text{coh}(\mathbb{P}(S^2W^\vee), \mathcal{C}_0)) = \langle \mathcal{C}_{1-n^2}, \mathcal{C}_{2-n^2}, \dots, \mathcal{C}_{-1}, \mathcal{C}_0 \rangle,$$

where

$$\mathcal{C}_1 = W \otimes \mathcal{O}_{\mathbb{P}(S^2W^\vee)} \oplus \Lambda^3 W \otimes \mathcal{O}_{\mathbb{P}(S^2W^\vee)}(-1) \oplus \dots$$

is the odd part of the Clifford algebra and  $\mathcal{C}_{k-2} = \mathcal{C}_k \otimes \mathcal{O}_{\mathbb{P}(S^2W^\vee)}(-1)$  for each  $k \in \mathbb{Z}$ . The picture visualizing this duality is:



For even  $n$  it has  $n/2$  gray columns and  $n^2/2$  white columns, and for odd  $n$  it has  $(n - 1)/2$  gray columns, one mixed column, and  $(n^2 - 1)/2$  white columns.

**5.4. Grassmannian–Pfaffian duality.** The most interesting series of examples is provided by Grassmannians  $\text{Gr}(2, m)$  of two-dimensional subspaces in an  $m$ -dimensional vector space.

Let  $W$  be a vector space of dimension  $m$  and let  $V = \Lambda^2 W$ , the space of bivectors. The group  $\text{GL}(W)$  acts on the projective space  $\mathbb{P}(\Lambda^2 W)$  with orbits indexed by the rank of a bivector which is always even and ranges from 2 to  $2\lfloor m/2 \rfloor$ . We denote by  $\text{Pf}(2k, W)$  the closure of the orbit consisting of bivectors of rank  $2k$  and call it the  $k$ -th Pfaffian variety. Clearly, the smallest orbit  $\text{Pf}(2, W)$  is smooth and coincides with the Grassmannian  $\text{Gr}(2, W)$  in its Plücker embedding. Another smooth Pfaffian variety is the maximal one —  $\text{Pf}(2\lfloor m/2 \rfloor, W) = \mathbb{P}(\Lambda^2 W)$ . All the intermediate Pfaffians are singular with  $\text{sing}(\text{Pf}(2k, W)) = \text{Pf}(2k - 2, W)$ . The submaximal Pfaffian variety  $\text{Pf}(2\lfloor m/2 \rfloor - 2, W^\vee)$  of the dual space is classically projectively dual to the Grassmannian  $\text{Gr}(2, W)$ . This suggests a possible HP duality between them.

To make a precise statement we should choose a Lefschetz decomposition of  $\mathbf{D}^b(\text{coh}(\text{Gr}(2, W)))$ . A naive choice is to take Kapranov’s collection (2.7). It can be considered as a Lefschetz decomposition on  $X := \text{Gr}(2, m)$  with  $m - 1$  blocks

$$\mathcal{A}_0 = \langle \mathcal{O}_X, \mathcal{U}_X^\vee, \dots, S^{m-2}\mathcal{U}_X^\vee \rangle, \mathcal{A}_1 = \langle \mathcal{O}_X, \mathcal{U}_X^\vee, \dots, S^{m-3}\mathcal{U}_X^\vee \rangle, \dots, \mathcal{A}_{m-2} = \langle \mathcal{O}_X \rangle.$$

However, it is very far from being minimal. It turns out that a reasonable result can be obtained for another Lefschetz decomposition

$$\mathbf{D}^b(\text{coh}(\text{Gr}(2, m))) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{m-1}(m - 1) \rangle$$

with

$$\mathcal{A}_0 = \dots = \mathcal{A}_{m-1} = \langle \mathcal{O}_X, \mathcal{U}_X^\vee, \dots, S^{(m-1)/2}\mathcal{U}_X^\vee \rangle. \tag{5.3}$$

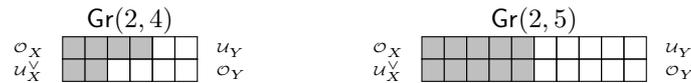
if  $m$  is odd, and with

$$\begin{aligned} \mathcal{A}_0 = \dots = \mathcal{A}_{m/2-1} &= \langle \mathcal{O}_X, \mathcal{U}_X^\vee, \dots, S^{m/2-1}\mathcal{U}_X^\vee \rangle, \\ \mathcal{A}_{m/2} = \dots = \mathcal{A}_{m-1} &= \langle \mathcal{O}_X, \mathcal{U}_X^\vee, \dots, S^{m/2-2}\mathcal{U}_X^\vee \rangle, \end{aligned} \tag{5.4}$$

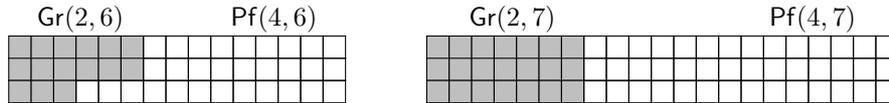
if  $m$  is even.

**Conjecture 5.4.** *The HP dual of the Grassmannian  $\text{Gr}(2, W)$  with Lefschetz decomposition (5.4) (or (5.3) depending on the parity of  $m = \dim W$ ) is given by a minimal categorical resolution of the submaximal Pfaffian  $\text{Pf}(2\lfloor m/2 \rfloor - 2, W^\vee)$ . When  $m$  is odd, this resolution is strongly crepant.*

This conjecture is proved for  $m \leq 7$  in [28]. In fact, for  $m = 2$  and  $m = 3$  one has  $\text{Gr}(2, W) = \mathbb{P}(\Lambda^2 W)$  and linear duality applies. For  $m = 4$  and  $m = 5$  the submaximal Pfaffian  $\text{Pf}(2, W^\vee)$  coincides with the Grassmannian, and the above duality is the duality for Grassmannians:



For  $m = 6$  and  $m = 7$  the submaximal Pfaffian  $Y = \text{Pf}(4, W^\vee)$  is singular, but its appropriate categorical resolutions can be constructed by Theorem 4.3. It turns out that these resolutions indeed are HP dual to the corresponding Grassmannians:



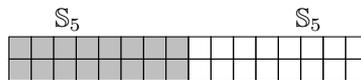
For  $m \geq 8$  this construction of a categorical resolution does not work. However it is plausible that the Pfaffians have wonderful resolutions of singularities, so a development of [1] may solve the question.

**5.5. The spinor duality.** Let  $W$  be a vector space of even dimension  $2m$  and  $q \in S^2 W^\vee$  a nondegenerate quadratic form. The isotropic Grassmannian of  $m$ -dimensional subspaces in  $W$  has two connected components, abstractly isomorphic to each other and called spinor varieties  $\mathbb{S}_m$ . These are homogeneous spaces of the spin group  $\text{Spin}(W)$  with the embedding into  $\mathbb{P}(\Lambda^m W)$  given by the square of the generator of the Picard group, while the generator itself gives an embedding into the projectivization  $\mathbb{P}(V)$  of a half-spinor representation  $V$  (of dimension  $2^{m-1}$ ) of  $\text{Spin}(W)$ . For small  $m$  the spinor varieties are very simple (because the spin-group simplifies): in fact,  $\mathbb{S}_1$  is a point,  $\mathbb{S}_2 = \mathbb{P}^1$ ,  $\mathbb{S}_3 = \mathbb{P}^3$ , and  $\mathbb{S}_4 = Q^6$ . The first interesting example is  $\mathbb{S}_5$ .

**Theorem 5.5** ([27]). *The spinor variety  $X = \mathbb{S}_5$  has a Lefschetz decomposition*

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_7(7) \rangle \quad \text{with} \quad \mathcal{A}_0 = \dots = \mathcal{A}_7 = \langle \mathcal{O}_X, \mathcal{U}_5^\vee \rangle. \quad (5.5)$$

*The HP dual variety is the same spinor variety  $Y = \mathbb{S}_5$ .*



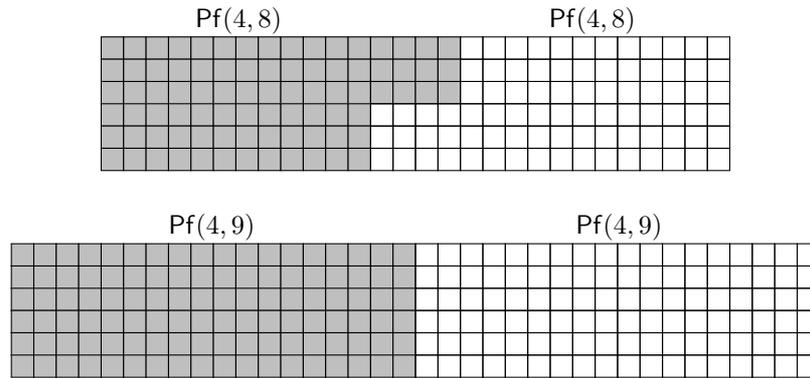
**5.6. Incomplete dualities.** It is often quite hard to give a full description of the HP dual variety. On the other hand, there is sometimes an open dense subset  $U \subset \mathbb{P}(V^\vee)$  for which there is a description of the category  $\mathcal{C}_U$  obtained by a base change  $U \rightarrow \mathbb{P}(V^\vee)$  from the HP dual category  $\mathcal{C}$ . If  $\mathbf{D}^b(\text{coh}(Y_U))$  is a (noncommutative) variety such that  $Y_U \cong \mathcal{C}_U$  (a  $U$ -linear equivalence), we will say that  $Y_U$  is the HP dual of  $X$  over  $U$ , or an incomplete HP dual variety.



**5.7. Conjectures.** Note that  $\text{Gr}(2, W) = \text{Pf}(2, W)$ . This suggests a generalization of the Grassmannian–Pfaffian duality to higher Pfaffians.

**Conjecture 5.9.** *For any  $k$  there is an HP duality between appropriate minimal categorical resolutions of the Pfaffians  $\text{Pf}(2k, W)$  and  $\text{Pf}(2(\lfloor m/2 \rfloor - k), W^\vee)$ . When  $m = \dim W$  is odd these resolutions are strongly crepant.*

Below are the expected pictures for the HP duality for  $\text{Pf}(4, 8)$  and for  $\text{Pf}(4, 9)$ :



It is also expected that there is a generalization of the Veronese–Clifford duality. Consider a vector space  $W$  of dimension  $n$  and its symmetric square  $V = S^2W$ . The group  $\text{GL}(W)$  acts on the projective space  $\mathbb{P}(V) = \mathbb{P}(S^2W)$  with orbits indexed by the rank of a tensor. We denote by  $\Sigma(k, W) \subset \mathbb{P}(S^2W)$  the closure of the orbit consisting of symmetric tensors of rank  $k$ . The smallest orbit  $\Sigma(1, W)$  is smooth and coincides with the double Veronese embedding of  $\mathbb{P}(W)$ . On the other hand,  $\Sigma(n, W) = \mathbb{P}(S^2W)$  is also smooth. All the intermediate varieties  $\Sigma(k, W)$  are singular with  $\text{sing}(\Sigma(k, W)) = \Sigma(k - 1, W)$ . The classical projective duality acts on these varieties by  $\Sigma(k, W)^\vee = \Sigma(n - k, W^\vee)$ . However, the HP duality is organized in a much more complicated way.

Besides  $\Sigma(k, W)$  itself one can consider its modifications:

- the Clifford modification  $(\Sigma(k, W), \mathcal{C}_0)$  for the natural sheaf of even parts of Clifford algebras on it, and
- (for even  $k$ ) the double covering  $\tilde{\Sigma}(k, W)$  of  $\Sigma(k, W)$  corresponding to the central subalgebra in  $\mathcal{C}_0$  as in (2.14),

and their minimal categorical resolutions. It seems that HP duality interchanges in a complicated way modifications of different type. For example, besides the original Veronese–Clifford duality between  $\Sigma(1, n)$  and  $(\Sigma(n, n), \mathcal{C}_0)$  there are strong indications that (the minimal resolution of)  $\Sigma(2, 4)$  is HP dual to (the minimal resolution of) the double covering  $\tilde{\Sigma}(4, 4)$  of  $\Sigma(4, 4) = \mathbb{P}^9$  (see [15]), (the minimal resolution of)  $\Sigma(2, 5)$  is HP dual to (the minimal resolution of) the double covering  $\tilde{\Sigma}(4, 5)$  of  $\Sigma(4, 5)$  (see [14]), while (the minimal resolution of) the double covering  $\tilde{\Sigma}(2, n)$  of  $\Sigma(2, n)$  is HP dual to (the minimal resolution of)  $\Sigma(n - 1, n)$  for all  $n$  (this can be easily deduced from the linear duality).

## 6. Varieties of small dimension

Let us list what is known about semiorthogonal decompositions of smooth projective varieties by dimension. In this section we assume that  $k = \mathbb{C}$ .

**6.1. Curves.** Curves are known to have no nontrivial semiorthogonal decompositions with the only exception of  $\mathbb{P}^1$  (for which every semiorthogonal decomposition coincides with the Beilinson decomposition up to a twist), see [44].

**6.2. Surfaces.** For surfaces the situation is more complicated. Of course, by the blowup formula any surface has a semiorthogonal decomposition with several exceptional objects and the derived category of a minimal surface as components. In particular, any rational surface has a full exceptional collection. Moreover, for  $\mathbb{P}^2$  it is known that any full exceptional collection can be obtained from Beilinson's collection by mutations (which are related to Markov numbers and toric degenerations of  $\mathbb{P}^2$ ). For other del Pezzo surfaces all exceptional objects have been classified [23], and moreover, three-blocks exceptional collections were constructed [18], but the complete picture is not known.

For minimal ruled surfaces, of course there is a semiorthogonal decomposition into two copies of the derived category of the curve which is the base of the ruling.

For surfaces of Kodaira dimension 0 it is well known that there are no nontrivial semiorthogonal decompositions for K3 and abelian surfaces. The same was proved for bielliptic surfaces in [21]. For Enriques surfaces there may be an exceptional collection of line bundles of length up to 10 (see [49]), and for so-called nodal Enriques surfaces the complementary component is related to the Artin–Mumford quartic double solid [15].

For surfaces of Kodaira dimension 1 with  $p_q > 0$  there are no semiorthogonal decompositions by [21].

Finally, for surfaces of general type there is an unexpectedly rich theory of semiorthogonal decompositions. In fact, for many surfaces of general type with  $p_g = q = 0$  (the classical Godeaux surface, the Beauville surface, the Burniat surfaces, the determinantal Barlow surface, some fake projective planes) exceptional collections of length equal to the rank of the Grothendieck group have been constructed in [2, 4, 5, 10, 12, 13]. The collections, however, are not full. The complementary components have finite (or even zero) Grothendieck group and trivial Hochschild homology and by that reason they are called quasiphantom or phantom categories. The phantoms cannot be detected by additive invariants, but one can use Hochschild cohomology instead, see [38].

An interesting feature here is that the structure of the constructed exceptional collections resembles very much the structure of exceptional collections of del Pezzo surfaces with the same  $K^2$ . The only (but a very important) difference is that whenever there is a Hom-space between exceptional bundles on del Pezzo, the corresponding exceptional bundles on the surface of general type have  $\text{Ext}^2$ -space. This seemingly small difference, however, has a very strong effect on the properties of the category. See more details in *loc. cit.*

**6.3. Fano 3-folds.** For derived categories of threefolds (and higher dimensional varieties) there are no classification results (as there is no classification of threefolds). Of course, as it already was mentioned for varieties with trivial (or globally generated) canonical class there are no nontrivial decompositions. So, from now on we will discuss Fano varieties.

In dimension 3 all Fano varieties were classified in the works of Fano, Iskovskikh and Mukai. All Fano 3-folds with Picard number greater than 1 are either the blowups of other Fano varieties with centers in points and smooth curves (and then their derived category reduces to the derived category of a Fano 3-fold with smaller Picard number), or conic bundles over rational surfaces (see Tables 12.3–12.6 of [17]). For conic bundles one can use the quadratic bundle formula (Theorem 2.7). It gives a semiorthogonal decomposition with several exceptional objects and the derived category of sheaves of modules over the even part of the Clifford algebra on the base of the bundle.

If the Picard number is 1, the next discrete invariant of a Fano 3-fold to look at is the index, i.e. the maximal integer dividing the canonical class. By Fujita's Theorem the only Fano 3-folds of index greater than 2 are  $\mathbb{P}^3$  and  $Q^3$ . Their derived categories are well understood, so let us turn to 3-folds of index 2 and 1.

For a Fano 3-fold  $Y$  of index 2 the pair of line bundles  $(\mathcal{O}_Y, \mathcal{O}_Y(1))$  is exceptional and gives rise to a semiorthogonal decomposition

$$\mathbf{D}^b(\mathrm{coh}(Y)) = \langle \mathcal{B}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle. \quad (6.1)$$

The component  $\mathcal{B}_Y$  is called the nontrivial component of  $\mathbf{D}^b(\mathrm{coh}(Y))$ .

A similar decomposition can be found for a Fano 3-fold  $X$  of index 1 if its degree  $d_X := (-K_X)^3$  is not divisible by 4 (the degree of a 3-fold of index 1 is always even). By a result of Mukai [43] if  $d_X > 2$  on such  $X$  there is an exceptional vector bundle  $\mathcal{E}_X$  of rank 2 with  $c_1(\mathcal{E}_X) = K_X$ , which is moreover orthogonal to the structure sheaf of  $X$ . In other words,  $(\mathcal{E}_X, \mathcal{O}_X)$  is an exceptional pair and there is a semiorthogonal decomposition

$$\mathbf{D}^b(\mathrm{coh}(X)) = \langle \mathcal{A}_X, \mathcal{E}_X, \mathcal{O}_X \rangle. \quad (6.2)$$

The component  $\mathcal{A}_X$  is called the nontrivial component of  $\mathbf{D}^b(\mathrm{coh}(X))$ .

It is rather unexpected that the nontrivial parts  $\mathcal{B}_Y$  and  $\mathcal{A}_X$  for a Fano 3-fold  $Y$  of index 2 and degree  $d_Y := (-K_Y/2)^3$  and for a Fano 3-fold  $X$  of index 1 and degree  $d_X = 4d_Y + 2$  have the same numerical characteristics, and are, moreover, expected to belong to the same deformation family of categories. In fact, this expectation is supported by the following result. Recall that the degree of a Fano 3-fold of index 2 with Picard number 1 satisfies  $1 \leq d \leq 5$ , while the degree of a Fano 3-fold of index 1 with Picard number 1 is even and satisfies  $2 \leq d \leq 22$ ,  $d \neq 20$ . So there are actually 5 cases to consider.

**Theorem 6.1** ([34]). *For  $3 \leq d \leq 5$  each category  $\mathcal{B}_{Y_d}$  is equivalent to some category  $\mathcal{A}_{X_{4d+2}}$  and vice versa.*

See *loc. cit.* for a precise statement. In fact, for  $d = 5$  the category is rigid and is equivalent to the derived category of representations of the quiver with 2 vertices and 3 arrows from the first vertex to the second (this follows from the construction of explicit exceptional collections in the derived categories of  $Y_5$  and  $X_{22}$ , see [45] and [24]). Further, for  $d = 4$  each of the categories  $\mathcal{B}_{Y_4}$  and  $\mathcal{A}_{X_{18}}$  is equivalent to the derived category of a curve of genus 2, and moreover, each smooth curve appears in both pictures. This follows from HP duality for the double Veronese embedding of  $\mathbb{P}^5$  and from HP duality for  $G_2$  Grassmannian respectively (Theorem 5.8). Finally, for  $d = 3$  no independent description of the category in question is known, but the HP duality for the Grassmannian  $\mathrm{Gr}(2, 6)$  gives the desired equivalence (see [25, 28]).

It turns out, however, that already for  $d = 2$  the situation is more subtle. It seems that in that case the categories  $\mathcal{B}_{Y_2}$  lie at the boundary of the family of categories  $\mathcal{A}_{X_{10}}$ . And for  $d = 1$  the situation is completely unclear.

The situation with Fano 3-folds of index 1 and degree divisible by 4 is somewhat different. For such threefolds it is, in general, not clear how one can construct an exceptional pair. However, for  $d_X = 12$  and  $d_X = 16$  this is possible. For  $d_X = 12$  Mukai has proved [43] that there is an exceptional pair  $(\mathcal{E}_5, \mathcal{O}_X)$  where  $\mathcal{E}_5$  is a rank 5 exceptional bundle with  $c_1(\mathcal{E}_5) = 2K_X$ . Using HP duality for the spinor variety  $\mathbb{S}_5$  (Theorem 5.5) one can check that this pair extends to a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(X_{12})) = \langle \mathbf{D}^b(\text{coh}(C_7)), \mathcal{E}_5, \mathcal{O}_{X_{12}} \rangle,$$

where  $C_7$  is a smooth curve of genus 7 (see [26, 27]). Analogously, for  $d_X = 16$  Mukai has constructed [43] an exceptional bundle  $\mathcal{E}_3$  of rank 3 with  $c_1(\mathcal{E}_3) = K_X$ . Using HP duality for  $\text{LGr}(3, 6)$  (Theorem 5.7) one can check that there is a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(X_{16})) = \langle \mathbf{D}^b(\text{coh}(C_3)), \mathcal{E}_3, \mathcal{O}_{X_{16}} \rangle,$$

where  $C_3$  is a smooth curve of genus 3 [27].

**6.4. Fourfolds.** Of course, for Fano 4-folds we know much less than for 3-folds. So, we will not even try to pursue a classification, but will restrict attention to some very special cases of interest.

Maybe one of the most interesting 4-folds is the cubic 4-fold. One of its salient features is the hyperkähler structure on the Fano scheme of lines, which turns out to be a deformation of the second Hilbert scheme of a K3 surface. This phenomenon has a nice explanation from the derived categories point of view.

**Theorem 6.2** ([36]). *Let  $Y \subset \mathbb{P}^5$  be a cubic 4-fold. Then there is a semiorthogonal decomposition*

$$\mathbf{D}^b(\text{coh}(Y)) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle,$$

*and its nontrivial component  $\mathcal{A}_Y$  is a Calabi–Yau category of dimension 2. Moreover, the category  $\mathcal{A}_Y$  is equivalent to the derived category of coherent sheaves on a K3 surface, at least if  $Y$  is a Pfaffian cubic 4-fold, or if  $Y$  contains a plane  $\Pi$  and a 2-cycle  $Z$  such that  $\deg Z + Z \cdot \Pi \equiv 1 \pmod{2}$ .*

To establish this result for Pfaffian cubics one can use HP duality for  $\text{Gr}(2, 6)$ . The associated K3 is then a linear section of this Grassmannian. For cubics with a plane a quadratic bundle formula for the projection of  $Y$  from the plane  $\Pi$  gives the result. The K3 surface then is the double covering of  $\mathbb{P}^2$  ramified in a sextic curve, and the cycle  $Z$  gives a splitting of the requisite Azumaya algebra on this K3.

For generic  $Y$  the category  $\mathcal{A}_Y$  can be thought of as the derived category of coherent sheaves on a noncommutative K3 surface. Therefore, any smooth moduli space of objects in  $\mathcal{A}_Y$  should be hyperkähler, and the Fano scheme of lines can be realized in this way, see [40].

The fact that a cubic 4-fold has something in common with a K3 surface can be easily seen from its Hodge diamond. In fact, the Hodge diamond of  $Y$  is

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 0 & 0 \\
 & & & & & 0 & 1 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 1 & 21 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 1 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 1 & & & & 
 \end{array}$$

and one sees immediately the Hodge diamond of a K3 surface in the primitive part of the cohomology of  $Y$ . There are some other 4-dimensional Fano varieties with a similar Hodge diamond. The simplest example is the 4-fold of degree 10 in  $\text{Gr}(2, 5)$  (an intersection of  $\text{Gr}(2, 5)$  with a hyperplane and a quadric in  $\mathbb{P}^9$ ). Its Hodge diamond is

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 0 & 0 \\
 & & & & & 0 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 1 & 22 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 1 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 1 & & & & 
 \end{array}$$

and again its primitive part has K3 type. On a categorical level this follows from the following result

**Theorem 6.3** ([35]). *Let  $X$  be a smooth projective variety of index  $m$  with a rectangular Lefschetz decomposition*

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{B}, \mathcal{B}(1), \dots, \mathcal{B}(m - 1) \rangle$$

of length  $m$ . Let  $Y_d$  be the smooth zero locus of a global section of the line bundle  $\mathcal{O}_X(d)$  for  $1 \leq d \leq m$ . Then there is a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(Y_d)) = \langle \mathcal{A}_{Y_d}, \mathcal{B}, \mathcal{B}(1), \dots, \mathcal{B}(m - d - 1) \rangle$$

and moreover, a power of the Serre functor  $\mathbf{S}_{\mathcal{A}_{Y_d}}$  is isomorphic to a shift

$$(\mathbf{S}_{\mathcal{A}_{Y_d}})^{d/c} = \left[ \frac{d \cdot (\dim X + 1) - 2m}{c} \right], \quad \text{where } c = \gcd(d, m). \tag{6.3}$$

In particular, if  $d$  divides  $m$  then the component  $\mathcal{A}_{Y_d}$  is a Calabi–Yau category of dimension  $\dim X + 1 - 2m/d$ .

**Remark 6.4.** Analogously, one can consider a double covering  $Y'_d \rightarrow X$  ramified in a zero locus of a global section of the line bundle  $\mathcal{O}_X(2d)$  instead. Then there is an analogous semiorthogonal decomposition and the Serre functor has the property

$$(\mathbf{S}_{\mathcal{A}_{Y'_d}})^{d/c} = \tau_*^{(m-d)/c} \circ \left[ \frac{d \cdot (\dim X + 1) - m}{c} \right], \tag{6.4}$$

where  $\tau$  is the involution of the double covering.

Applying this result to a 4-fold  $Y$  of degree 10 one constructs a semiorthogonal decomposition  $\mathbf{D}^b(\text{coh}(Y)) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{U}_Y^\vee, \mathcal{O}_Y(1), \mathcal{U}_Y^\vee(1) \rangle$  with  $\mathcal{U}_Y$  being the restriction of the tautological bundle from the Grassmannian  $\text{Gr}(2, 5)$  and  $\mathcal{A}_Y$  a Calabi–Yau category of dimension 2. Again, for some special  $Y$  one can check that  $\mathcal{A}_Y$  is equivalent to the derived category of a K3 surface and so altogether we get another family of noncommutative K3 categories. Moreover, in this case one can also construct a hyperkähler fourfold from  $Y$ . One of the ways is to consider the Fano scheme of conics on  $Y$ . It was proved in [16] that it comes with a morphism to  $\mathbb{P}^5$  with the image being a singular sextic hypersurface, and the Stein factorization of this map gives a genus zero fibration over the double covering of the sextic, known as a double EPW sextic. This is a hyperkähler variety, deformation equivalent to the second Hilbert square of a K3 surface.

Finally, there is yet another interesting example. Consider a hyperplane section  $Y$  of a 5-fold  $X$ , which is the zero locus of a global section of the vector bundle  $\Lambda^2 \mathcal{U}_3^\vee \oplus \Lambda^3(W/\mathcal{U}_3)$  on  $\text{Gr}(3, W)$  with  $W$  of dimension 7. This variety  $Y$  was found by O. Küchle in [22] (variety c5 in his table), and its Hodge diamond is as follows

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & 0 & 0 \\
 & & & 0 & 1 & 0 \\
 & & 0 & 0 & 0 & 0 \\
 0 & & 1 & 24 & 1 & 0 \\
 & & 0 & 0 & 0 & 0 \\
 & & 0 & 1 & 0 & \\
 & & 0 & 0 & & \\
 & & & & & 1
 \end{array}$$

**Conjecture 6.5.** *The 5-dimensional variety  $X \subset \text{Gr}(3, W)$  has a rectangular Lefschetz decomposition  $\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{B}, \mathcal{B}(1) \rangle$  with the category  $\mathcal{B}$  generated by 6 exceptional objects. Consequently, its hyperplane section  $Y$  has a semiorthogonal decomposition  $\mathbf{D}^b(\text{coh}(Y)) = \langle \mathcal{A}_Y, \mathcal{B} \rangle$  with  $\mathcal{A}_Y$  being a K3 type category.*

It would be very interesting to understand the geometry of this variety and to find out, whether there is a hyperkähler variety associated to it, analogous to the Fano scheme of lines on a cubic fourfold and the double EPW sextic associated to the 4-fold of degree 10. A natural candidate is the moduli space of twisted cubic curves.

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