Semiorthogonal Decomposition of Derived Categories | Introductory School on Derived Categories

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Lecture 1. | University of Warwick.

A triangulated category $\mathcal{T}$ is an additive category (k-linear). This comes with

- Shift functors: $[t] : \mathcal{T} \to \mathcal{T}$ for each $t \in \mathbb{Z}$ (Note $[t_1]\cdot[t_2] = [t_1+t_2]$).
  - better to have equality and not $\simeq$. Can always pass to an equiv. category to make this equality.
- Class of distinguished triangles

$$T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} T_3 \xrightarrow{f_3} T_1[1].$$

The reader is referred to Gelfand-Manin for all axioms of triangulated categories. The most important axiom is:

For any $f: T_1 \to T_2$, $\exists$ a distinguished triangle $T_1 \xrightarrow{f} T_2 \longrightarrow T_3 \longrightarrow T_1[1]$

$T_3 =: \mathrm{Cone}(f)$.

The isomorphism class of $\mathrm{Cone}(f)$ is well defined but is not functorial.

Some remarks about

$$\begin{tikzcd}
T_1' \arrow[r] \arrow[d] & T_2' \arrow[r] \arrow[d] & T_3' \arrow[r] \arrow[d, squiggly] & T_1'[1] \\
T_1 \arrow[r, "f"] & T_2 \arrow[r] & T_3 \arrow[r] & T_1[1]
\end{tikzcd}$$

Some notation:

- $D(\mathcal{C})$:= the derived category of the abelian category $\mathcal{C}$.
- $D^b(\mathcal{C})$ = Bounded derived category.
- $\mathrm{Ext}^i(T,T') = \mathrm{Hom}^i(T,T') = \mathrm{Hom}(T,T'[i])$.
- $\mathrm{Hom}^\bullet(T,T') = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(T,T')[-i] \in \mathrm{grmod}(k) = D(k)$.
- $V \in \mathrm{mod}(k)$; $T \in \mathcal{T}$ then $V \otimes_k T = T^{\oplus \dim V}$

  This is not a good definition (non-functorial)

  So you define it by the functor it represents

$$\left.\begin{aligned} \mathrm{Hom}(V\otimes T, T') &= \mathrm{Hom}(V^*, \mathrm{Hom}(T,T')) \\ \mathrm{Hom}(T', V\otimes T) &= V \otimes \mathrm{Hom}(T',T). \end{aligned}\right\} (*)$$

- $V^\bullet \in \mathrm{grmod}(k)$; $V^\bullet = \bigoplus_{i\in\mathbb{Z}} V_i[-i]$ (Replace Hom with $\mathrm{Hom}^\bullet$

$$V^\bullet \otimes T = \bigoplus_{i\in\mathbb{Z}} V_i \otimes T[-i].$$

and V with $V^\bullet$ in the above conditions $(*)$)

The most important notion for these lectures is that of:

Semiorthogonal Decomposition.

Let $\mathcal{T}$ be a triangulated category, $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ be strictly full triangulated subcategories

is closed under shifts and cones.

Def: $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition if

Can also take $\mathrm{Hom}^\bullet$

1) $\mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$ (ie $\forall A \in \mathcal{A}$, $B \in \mathcal{B}$ we have $\mathrm{Hom}(B, A) = 0$).

2) For any $T \in \mathcal{T}$ there is a distinguished triangle $T_{\mathcal{B}} \to T \to T_{\mathcal{A}} \to T_{\mathcal{B}}[1]$, with $T_{\mathcal{A}} \in \mathcal{A}$ and $T_{\mathcal{B}} \in \mathcal{B}$.

Properties:

1) Functoriality: For any $f: T \to T'$ there exist unique $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ such that the following diagram commutes:

in proof

map below is just composition with this arrow

$$\begin{tikzcd} T_{\mathcal{B}} \arrow[r] \arrow[d, dashed, "f_{\mathcal{B}}"'] & T \arrow[r] \arrow[d, "f"] & T_{\mathcal{A}} \arrow[r] \arrow[d, dashed, "f_{\mathcal{A}}"] & T_{\mathcal{B}}[1] \arrow[d, dashed, "f_{\mathcal{B}}[1]"] \\ T'_{\mathcal{B}} \arrow[r] & T' \arrow[r] & T'_{\mathcal{A}} \arrow[r] & T'_{\mathcal{B}}[1] \end{tikzcd}$$

comments in proof

Proof: Apply $\mathrm{Hom}(-, T'_{\mathcal{A}})$. Get long exact sequence)

$$\dots \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}(T_{\mathcal{B}}[1], T'_{\mathcal{A}})} \to \underset{\substack{\cup \\ f_{\mathcal{A}}}}{\mathrm{Hom}(T_{\mathcal{A}}, T'_{\mathcal{A}})} \xrightarrow{\cong} \mathrm{Hom}(T, T'_{\mathcal{A}}) \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}(T_{\mathcal{B}}, T'_{\mathcal{A}})} \to \dots$$

$\Rightarrow \exists!\ f_{\mathcal{A}}$ s.t the middle square commutes.

Exercise: $\exists!\ f_{\mathcal{B}}$ s.t the left square commutes. $\square$

The exercise shows that the left square commutes.

Corollary:

$\left.\begin{array}{l} T \mapsto T_{\mathcal{A}} \\ f \mapsto f_{\mathcal{A}} \end{array}\right\}$ is a functor and so is $\begin{array}{l} T \mapsto T_{\mathcal{B}} \\ f \mapsto f_{\mathcal{B}} \end{array}$.

These are both functors from $\mathcal{T} \to \mathcal{A}$ and $\mathcal{T} \to \mathcal{B}$.

Let the adjoint functors be:

$\mathcal{A} \xrightarrow{\alpha} \mathcal{T}$ and $\mathcal{B} \xrightarrow{\beta} \mathcal{T}$.

Then $T \mapsto T_{\mathcal{A}}$ is left adjoint to α (denoted α^*) and $T \mapsto T_{\mathcal{B}}$ is right adjoint to β (denoted $\beta^!$)