INTRODUCTION TO NONCOMMUTATIVE GEOMETRY

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FOREWORD

The roots of noncommutative geometry lie in the theory of commutative Banach algebras and its connections with topology established by I.M.Gelhand, M.A.Naimark, G.E.Shilov, B.Mazur and other mathematicians in the middle of XXth century. Their main idea was that basic notions of the topology of compact topological spaces may be translated into the language of commutative Banach algebras of continuous functions on these compacta.

One of the goals of noncommutative geometry is to establish such correspondence between topology, analysis and differential geometry, on one hand and Banach algebras, on the other hand. Otherwise speaking, we would like to translate basic notions of topology, analysis and geometry into the language of Banach algebras. In this case we cannot restrict any longer to the theory of commutative Banach algebras but have to use also noncommutative Banach algebras, more precisely the C^*-algebras of operators in a Hilbert space.

At that point a natural question arises: why do we need such a translation? One of the motivations behind it comes from theoretical physics. It is well known that the quantum field theory remains to a large extent the physical theory without a solid mathematical basis. In contrast with quantum mechanics, which may be considered with some caution as a mathematically rigorous discipline, many results of quantum field theory (and string theory, in particular) are established only on the "physical level of rigorousness" and do not have correct mathematical proofs. One of the reasons for such a situation, in our opinion, is the absence of the adequate mathematical language adjusted for the description of the arising physical problems.

We think that such language should incorporate as one of its important ingredients the well-developed differential geometry of smooth infinite-dimensional manifolds. However, the classical notions of differential geometry such as connection, curvature et al. do not survive after transition to infinite dimensions. For example, various equivalent definitions of connection, known in the finite-dimensional case, acquire completely different sense (or loose it at all) for infinite-dimensional manifolds. It is even more true with respect to the curvature which in most cases cannot be correctly defined by analogy with the finite-dimensional case.

In this situation it seems that the most adequate language for the description of notions of topology, analysis and differential geometry on infinite-dimensional manifolds should be the most "robust" among available mathematical languages, namely, the algebraic one. This language has the highest chances to "survive" under transition to infinite dimensions. To make this language ready for use in the infinite-dimensional situation we should have available a "dictionary" translating basic notions of topology, analysis and differential geometry into the algebraic language in the usual finite-dimensional setting. The elaboration of such dictionary is precisely the main goal of this lecture course. Having it in mind, we concentrate here mostly on the ideas rather than technical details. By the same reason we omit some proofs providing instead the references to other books.

This text is based on the lecture course delivered by the author at the Scientific Educational Center (SEC) of Steklov Institute during the spring semesters of years 2014 and 2015.

The lectures were accompanied by the seminar of SEC on the same topic. Many problems which are only touched upon in this text were considered in detail at the seminar. I am grateful to Alexander Komlov, Innocenti Maresin and Roman Palvelev who carried out the main burden of organizing this seminar. I would also like to thank the listeners of the course who helped a lot to improve the original text of the lectures with their questions and remarks.

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Chapter 1

TOPOLOGY

1.1 Commutative Banach algebras

1.1.1 C^*-algebras

Definition 1. The *Banach algebra* is an associative algebra A over the field $\mathbb{C}$ which is simultaneously a complete normed space with the norm satisfying the inequality:

$$\|ab\| \leq \|a\| \cdot \|b\|$$

for all elements $a, b \in A$. The algebra A is called *unital* if it contains the unit 1 with the norm $\|1\| = 1$.

Definition 2. The *involution* in A is an isometric antilinear map $a \mapsto a^*$ with the following properties:

$$a^{**} = a, \quad (ab)^* = b^* a^*$$

for any $a, b \in A$. A Banach algebra with involution is called otherwise the *Banach *-algebra*. The *C^*-algebra* is a Banach *-algebra A with the following additional property:

$$\|a^2\| = \|a^* a\|$$

for all $a \in A$.

The standard example of the C^*-algebra is the algebra of continuous functions on a compact.

Example 1. Let X be a compact topological space (assumed to be Hausdorff as all other topological spaces considered in this text). Denote by $C(X)$ the algebra of continuous functions $f : X \to \mathbb{C}$ provided with the norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

The unit in $C(X)$ coincides with the function $f \equiv 1$ and involution $f \mapsto f^*$ is given by the complex conjugation: $f^*(x) := \overline{f(x)}$. The introduced norm has the property

$$\|f\|^2 = \|f^* f\|$$

so $C(X)$ is a C^*-algebra. Summing up, $C(X)$ is a commutative unital C^*-algebra.

1.1.2 Adding of unit and compactification

Any non-unital Banach algebra A may be embedded into a unital one by adding formally the unit 1_A. In other words, we can extend A to the algebra $A^+ := A \times \mathbb{C}$ with evident rules of summation, multiplication by complex numbers and involution. The product in A^+ is defined by

$$(a, \lambda) \cdot (b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$$

so that 1_A is identified with the element $(0, 1)$. The norm of an element (a, λ) is defined by:

$$\|(a, \lambda)\| := \sup_{\|b\| \leq 1} \{\|ab + \lambda b\|\}.$$

The constructed algebra A^+ is a unital C^*-algebra if A is a C^*-algebra.

Consider this *unitalization* procedure in the case of the commutative Banach algebra of continuous functions.

Let Y be a locally compact topological space. Denote by $Y^+ := Y \cup \{\infty\}$ the one-point compactification of Y and consider the subalgebra of $C(Y^+)$ consisting of functions vanishing at infinity. By restricting the functions from this subalgebra to Y we can identify it with the subalgebra $C_0(Y)$ in $C(Y)$, consisting of the functions vanishing at infinity. By definition the elements of this subalgebra are the functions from $C(Y)$ which have the following property: the set $\{y \in Y : |f(y)| \geq \varepsilon\}$ is compact for any $\varepsilon > 0$. The unitalization of this subalgebra coincides with the algebra $C(Y^+)$.

Conversely, if we exclude from a compact topological space X a non-isolated point $x_0 \in X$ then the obtained space $Y := X \setminus x_0\}$ will be locally compact with $Y^+ = X$ and $C_0(Y) = \{f \in C(X) : f(x_0) = 0\}$.

So the described unitalization procedure corresponds in the language of topological spaces to the one-point compactification.

1.1.3 Characters and spectrum

Definition 3. A *character* of a Banach algebra A is an algebra homomorphism $\mu : A \to \mathbb{C}$. In other words, it is a non-zero linear functional $\mu : A \to \mathbb{C}$ on the algebra A with the multiplicativity property, i.e. $\mu(ab) = \mu(a)\mu(b)$ for all $a, b \in A$. If the algebra A is unital then $\mu(1_A) = 1$. The set of characters of the algebra A is denoted by $M(A)$ and called otherwise the *spectrum of the algebra A*.

Example 2. An example of a character for the algebra $A = C(X)$ of continuous functions on a compact topological space X is given by the *evaluation* at $x \in X$:

$$\varepsilon_x : f \longmapsto f(x), \quad f \in A.$$

Remark 1. In the case when A is a non-unital algebra any character $\mu \in M(A)$ may be extended to its unitalization A^+ by setting: $\mu(0, 1) = 1$. Consider the character on A^+ determined by the formula: $(a, \lambda) \mapsto \lambda$. Its restriction to A yields the zero functional. So the space $M(A^+)$ may be identified with $M(A) \cup \{0\}$.

Definition 4. The *spectrum* $\mathrm{sp}(a)$ of an element a of a unital Banach algebra A is the set of complex numbers λ such that the element $a - \lambda 1_A$ is not invertible in A. If the algebra A is non-unital that the spectrum of a consists of complex numbers λ such that the element $a - \lambda 1_{A^+}$ is not invertible in A^+.

We formulate two important properties of the spectrum of an element of a unital Banach algebra A which are proved in the same way as the analogous properties of the spectrum of a bounded linear operator in a Hilbert space.

- The spectrum $\mathrm{sp}(a)$ of an arbitrary element $a \in A$ is a closed set. It is bounded and contained in the disk $\{\lambda \in \mathbb{C} : |\lambda| \le \|a\|\}$, in particular, $\mathrm{sp}(a)$ is a compact set.
- The spectrum $\mathrm{sp}(a)$ of an arbitrary element $a \in A$ is not empty.

For a unital algebra A the value $\mu(a)$ of any character $\mu \in M(A)$ at an arbitrary point $a \in A$ belongs to the spectrum: $\mu(a) \in \mathrm{sp}(a)$, since otherwise the element $\mu(a - \mu(a)1_A) = 0$ would be invertible in $\mathbb{C}$. Since $\mathrm{sp}(a) \subset \{\lambda \in \mathbb{C} : |\lambda| \le \|a\|\}$, it implies that $|\mu(a)| \le \|a\|$, i.e. $\|\mu\| \le 1$ where

$$\|\mu\| = \sup_{a \in A} \frac{|\mu(a)\|}{\|a\|} . \tag{1.1}$$

It follows that $\|\mu\| = 1$ since $\mu(1_A) = 1$.

Definition 5. An element $a \in A$ of a Banach *-algebra A is called *self-adjoint* if $a^* = a$.

Note that an arbitrary element a of a Banach algebra A may be written as the sum $a = \alpha + i\beta$ of elements of A given by

$$\alpha = \frac{a + a^*}{2} , \ \beta = \frac{a - a^*}{2i} \tag{1.2}$$

which are self-adjoint. It follows that $a^* = \alpha - i\beta$.

Lemma 1. *Let a be a self-adjoint element of a C^*-algebra A. Then the value $\mu(a)$ of any character $\mu \in M(A)$ at a is a real number.*

Proof. Consider the exponential $u := \exp(ia)$ of the element ia. It is given by the series

$$u = \sum_{k=0}^{\infty} \frac{(ia)^k}{k!}$$

which converges and satisfies the estimate: $\|u\| \le e^{\|a\|}$. Moreover, $u^* = \exp(-ia)$ so $uu^* = u^*u = 1_A$. It implies that the element u is convertible and $u^{-1} = u^*$. The relation $\|u\|^2 = \|u^*u\| = 1$ implies that $\|u\| = 1$ and, analogously, $\|u^{-1}\| = 1$. Since $\|\mu\| \le 1$ the two last equalities imply that $|\mu(u)| \le 1$ and $|\mu(u)|^{-1} = |\mu(u^{-1})| \le 1$ whence $|\mu(u)| = 1$. But

$$\mu(u) = \sum_{k=0}^{\infty} \frac{\mu(ia)^k}{k!} = e^{i\mu(a)}.$$

Since $|\mu(u)| = 1$, it is possible only if $\mu(a) \in \mathbb{R}$. $\square$

Introduce one more important notion related to the spectrum.

Definition 6. The *spectral radius* of an element $a \in A$ is the number $r(a)$ equal to

$$r(a) = \max\{|\lambda| : \lambda \in \mathrm{sp}(a)\}.$$

In other words, $r(a)$ coincides with the radius of the smallest closed disk with the center at zero containing $\mathrm{sp}(a)$. In particular, $r(a) \le \|a\|$.

The spectral radius may be computed by the *Cauchy–Hadamard formula*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

It implies that for a self-adjoint element $a = a^* \in A$ its spectral radius coincides with the norm: $r(a) = \|a\|$ (why?).

1.1.4 Spectrum of a commutative Banach algebra. Gelfand transform

Denote by A' the Banach space of continuous linear functionals $\varphi : A \to \mathbb{C}$ on the algebra A with the norm (1.1). Provide it with the weak-$*$ topology, i.e. topology of the uniform convergence on elements from A. In the case when $A = C(X)$ is the algebra of continuous functions on a compact X the space A' consists of complex measures on X with the usual topology. By the Banach–Alaoglu theorem the unit ball A'_1 in the space A' is compact in the weak-$*$ topology. Since the spectrum $M(A)$ of the algebra A is contained in A'_1 we can provide it with the induced topology.

Lemma 2. *The spectrum of a commutative Banach algebra is a locally compact topological space.*

Proof. We show first that the space $M(A) \cup \{0\}$ is closed in the weak-$*$ topology, i.e. that the weak-$*$ limit of elements from $M(A) \cup \{0\}$ belongs again to $M(A) \cup \{0\}$. Since the weak-$*$ limit of continuous linear functionals from A' belongs to A' we have to check only that the multiplicativity property is preserved under this limit. Indeed, for fixed elements $a, b \in A$ the map

$$M(A) \cup \{0\} \ni \mu \longmapsto \mu(a)\mu(b) - \mu(ab)$$

is weak-$*$ continuous and vanishes on $M(A) \cup \{0\}$, hence also in the closure of $M(A) \cup \{0\}$ in A'. This proves that the space $M(A) \cup \{0\}$ is closed. Hence, it is compact being a closed subset of the compact set A'_1. It follows that the space $M(A)$ is locally compact.

If the algebra A is unital then already the space $M(A)$ is compact since the point $\{0\}$ (corresponding to the evaluation at infinity) is isolated from $M(A)$ (note that the continuous functional $M(A) \ni \mu \mapsto \mu(1_A)$ separates it from $M(A)$). □

Definition 7. Let A be a commutative Banach algebra. Its *Gelfand transform* is the map

$$\mathcal{G} : A \longrightarrow C_0(M(A))$$

given by the formula

$$A \ni a \longmapsto \hat{a} : M(A) \to \mathbb{C} \quad \text{where } \hat{a}(\mu) := \mu(a).$$

We denote by $C_0(M(A))$ as above the space of continuous functions on $M(A)$ vanishing at infinity.

The Gelfand transform $A \to C_0(M(A))$ is evidently continuous. Let us consider its action on the involution in A. If A is a C^*-algebra and $\mu \in M(A)$ then, in accordance with decomposition (1.2),

$$\mu(a^*) = \mu(\alpha - i\beta) = \mu(\alpha) - i\mu(\beta) = \overline{\mu(a)}.$$

The last equality holds since by Lemma 1 the values of the character μ on self-adjoint elements are real. So, $\widehat{a^*}(\mu) = \overline{\hat{a}(\mu)}$, i.e. $\mathcal{G}(a^*) = \overline{\mathcal{G}(a)}$. In other words, Gelfand transform commutes with involutions in A and $C_0(M(A))$, i.e. it is a *-*homomorphism*.

For the characters the following analogue of the Hahn–Banach theorem holds.

Lemma 3. *Let A be a unital commutative C^*-algebra and $a \in A$. If $\lambda \in sp(a)$ then there exists a character $\mu \in M(A)$ such that $\mu(a) = \lambda$.*

Proof. We prove that the *kernel of any character of A is a maximal ideal and, conversely, any maximal ideal in A coincides with the kernel of some character.*

Assume that this fact is already established and deduce the assertion of our lemma from it. Consider the ideal of A of the form $(a - \lambda 1_A)A$. Then this ideal is contained in some maximal ideal (this assertion follows from the Zorn lemma; why?) which, according to the assumed fact, coincides with the kernel $\operatorname{Ker}\mu$ of some character μ. In other words, $\mu(a - \lambda 1_A) = 0$, i.e. $\mu(a) = \lambda$.

Return to the proof of the fact formulated above. If μ is a character of the algebra A then its kernel $\operatorname{Ker}\mu$ is an ideal in A. Suppose that this ideal is not maximal, i.e. there exists a proper ideal I of the algebra A which contains $\operatorname{Ker}\mu$ but does not coincide with it. Take an element $a_0 \in I$ which does not belong to $\operatorname{Ker}\mu$. Then the character μ does not vanish on this element. But the element $a_0 - \mu(a_0)1_A$ belongs to the kernel of μ, hence to the ideal I. So the element $a_0 - (a_0 - \mu(a_0)1_A) = \mu(a_0)1_A$ also belongs to the ideal I which implies that I contains 1_A, i.e. $I = A$, contradicting our assumption that it is proper.

Conversely, let I be a proper maximal ideal in A. Note first of all that I is closed. Indeed, the closure of I is an ideal in A which, due to the maximality of I should coincide either with the ideal I itself, or with the whole algebra A. But a proper ideal cannot be dense in A since the set of invertible elements of the algebra A is, evidently, open and the ideal I does not contain any of them (otherwise, it would coincide with A). Hence, the closure of I should coincide with I, i.e. ideal I is closed.

Since I is closed the quotient A/I modulo this ideal is a commutative Banach algebra with unit. Moreover, it is simple, i.e. does not contain any proper ideals. Indeed, if such an ideal would exist then its preimage under the natural projection $A \to A/I$ would be a proper ideal in A, containing I, which is impossible due to the maximality of I. It means that the quotient algebra A/I is a field, i.e. any nonzero element of this algebra is invertible (since the principal ideal generated by such element should coincide with the whole algebra).

This field coincides necessarily with $\mathbb{C}$. Indeed, take an arbitrary element $x \in A/I$. Since its spectrum $\mathrm{sp}(x)$ is not empty there exists a $\lambda \in \mathrm{sp}(x)$ such that the element $x - \lambda 1_A$ is not invertible, i.e. it should be equal to zero according to the above argument, whence $x = \lambda 1_A$. Thus, we have established an isomorphism between A/I and $\mathbb{C}$.

Having proved that $A/I = \mathbb{C}$ we deduce that the natural projection $A \to A/I$ is a character of the algebra A with the kernel equal to I. □

1.1.5 Gelfand–Naimark theorem

Before we formulate this theorem recall the Stone–Weierstrass theorem which will be used in the proof of Gelfand–Naimark theorem.

Suppose that X is a locally compact topological space and B is a subalgebra in $C_0(X)$. We say that B *does not vanish at* $x \in X$ if there exists a function from B which does not vanish at this point.

Theorem 1 (Stone–Weierstrass)**.** *Let X be a locally compact topological space and B is a closed subalgebra in $C_0(X)$ which separates the points of X. Suppose that B does not vanish at any point of X and is closed under the complex conjugation. Then $B = C_0(X)$.*

Theorem 2 (Gelfand–Naimark)**.** *Let A be a commutative C^*-algebra. Then the Gelfand transform establishes an isometric *-isomorphism $\mathcal{G} : A \to C_0(M(A))$.*

Proof. We have already proved that Gelfand transform is a *-homomorphism. Its isometricity follows from the following chain of equalities:

$$\|\widehat{a}\|^2 = \|\hat{a}^*\hat{a}\| = \|\widehat{a^*a}\| = r(a^*a) = \|a^*a\| = \|a\|^2$$

where the equality $\|\widehat{a^*a}\| = r(a^*a)$ (r is the spectral radius) follows from Lemma 3 (why?).

In particular, Gelfand transform is injective. Hence, the image $\mathcal{G}(A)$ of the algebra A in $C_0(M(A))$ is a subalgebra which is complete (since the algebra A is complete and the map $\mathcal{G}$ is isometric) hence, also closed. The evaluation map separates multiplicative functionals on A and the algebra $\mathcal{G}(A)$ does not vanish at any point of $M(A)$. Moreover, it is closed with respect to complex conjugation. Hence, by Stone–Weierstrass theorem the subalgebra $\mathcal{G}(A)$ coincides with $C_0(M(A))$. □

1.1.6 Positive elements and states

Definition 8. An element $a \in A$ of a C^*-algebra A is called *positive* if it is self-adjoint and its spectrum $\mathrm{sp}(a)$ is non-negative, i.e. belongs to $[0, \infty)$.

It is easy to show that an element $a \in A$ is positive if and only if it can be represented in the form $a = b^*b$ for some element $b \in A$. (To prove this assertion use the fact that any positive element has the square root, i.e. an element $\sqrt{a} \in A$ such that $(\sqrt{a})^2 = a$.)

Using Definition 8, we can introduce a partial ordering relation on self-adjoint elements of the C^*-algebra A.

Definition 9. Let a and b be self-adjoint elements of a C^*-algebra A. We say that $a \leqslant b$ if the element $b - a$ is positive.

Any positive element $a \in A$ satisfy the inequality

$$0 \leq a \leq \|a\| \, 1_A$$

since $\mathrm{sp}(\|a\| \, 1_A - a) \subset [0, +\infty)$.

Definition 10. A linear functional $\varphi : A \to \mathbb{C}$ is called *positive* if $\varphi(a) \geqslant 0$ for any positive element $a \in A$ or, equivalently, $\varphi(b^*b) \geq 0$ for any element $b \in A$.

If the algebra A is unital then any positive functional φ on this algebra is bounded and $\|\varphi\| = \varphi(1_A)$. Conversely, any linear bounded functional on such algebra with the property: $\|\varphi\| = \varphi(1_A)$, is necessarily positive.

Definition 11. A positive linear functional φ on a C^*-algebra A is called the *state* if its norm is equal to 1: $\|\varphi\| = 1$. The state is called the *trace* if $\varphi(ab) = \varphi(ba)$ for any $a, b \in A$.

The set of states is convex, i.e. for any states φ, ψ on the algebra A their convex linear combination $t\varphi + (1 - t)\psi$ with $t \in [0, 1]$ is also a state.

Definition 12. A state φ is called *pure* if it cannot be represented as a convex linear combination of other states, i.e. if φ cannot be represented as the sum $\varphi = t\psi_1 + (1 - t)\psi_2$ where ψ_1, ψ_2 are the states on A different from φ and $t \in (0, 1)$.

1.1.7 Gelfand–Naimark–Segal construction (GNS-construction)

The GNS-construction allows to construct from any state φ on a C^*-algebra A a *-representation π_φ of the algebra A in some Hilbert space $\mathcal{H}_\varphi$. This representation is given by a homomorphism of A into the algebra $\mathcal{L}(\mathcal{H}_\varphi)$ of bounded linear operators in $\mathcal{H}_\varphi$.

We turn to the detailed description of the *GNS-construction.*

Define a positive semi-definite (perhaps, degenerate) sesquilinear form on A by the formula

$$(a, b)_\varphi = \varphi(a^*b).$$

It is linear in the second argument and anti-linear in the first one. Moreover, this form satisfies the Cauchy inequality

$$|(a, b)_\varphi|^2 \leqslant (a, a)_\varphi (b, b)_\varphi.$$

The constructed form degenerates on elements of the left ideal

$$N_\varphi := \{a \in A : \ (a, a)_\varphi = 0\} = \{b \in A : \ \varphi(a^*b) = 0 \text{ for all } a \in A\}.$$

Consider the quotient A/N_φ consisting of the elements given by the classes $[a] := a + N_\varphi$. Define an inner product on this quotient by:

$$([a], [b])_\varphi := (a, b)_\varphi.$$

This inner product is correctly defined on A/N_φ and non-degenerate (check it!).

Introduce now the Hilbert space $\mathcal{H}_\varphi$ given by the completion of the space A/N_φ with respect to the norm associated with this inner product.

Define the representation π_φ of the algebra A in the Hilbert space $\mathcal{H}_\varphi$ by the formula:

$$\pi_\varphi(a) : [b] \longmapsto [ab].$$

The operator $\pi_\varphi(a)$ is bounded on A/N_φ and $\|\pi_\varphi(a)\| \leqslant \|a\|$. Moreover, the representation π_φ is a *-homomorphism, i.e. $\pi_\varphi(a^*) = (\pi_\varphi(a))^\dagger$ where $(\pi_\varphi(a))^\dagger$ is the Hermitian conjugate of $\pi_\varphi(a)$.

Since the operator $\pi_\varphi(a)$ is bounded on the subspace A/N_φ, which is dense in $\mathcal{H}_\varphi$, it can be extended to a bounded linear operator denoted by the same letter $\pi_\varphi(a)$ which is defined on the whole Hilbert space $\mathcal{H}_\varphi$ and satisfies the estimate: $\|\pi_\varphi(a)\| \leq \|a\|$.

Note that if the algebra A is unital then the vector $\xi := [1_A]$ is *cyclic* for the representation π_φ, i.e. the set of elements $\{\pi_\varphi(a)\xi : a \in A\}$ is dense in $\mathcal{H}_\varphi$. Moreover, for any $a \in A$ we have the equality

$$(\xi, \pi_\varphi(a)\xi)_\varphi = \varphi(a). \tag{1.3}$$

It can be shown that the constructed representation π_φ is irreducible if and only if the state φ is pure.

1.1.8 Embedding of C^*-algebras into the algebra of bounded linear operators in a Hilbert space

An arbitrary C^*-algebra may be embedded into the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators in some Hilbert space according to the following Gelfand–Naimark theorem.

Theorem 3 (Gelfand–Naimark)**.** *Any C^*-algebra A is isomorphic to a closed subalgebra in the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators acting in some Hilbert space $\mathcal{H}$.*

Idea of the proof. Using the Hahn–Banach theorem, it may be proved that for any element $a \in A \setminus \{0\}$ there exists a state φ_a such that

$$\varphi_a(a^*a) = \|a\|^2.$$

Then the equality (1.3) implies that $\|\pi_\varphi(a)\xi\|_\varphi = \|a\|$. It follows that $\pi_\varphi(a)$ can vanish only for $a = 0$.

Consider now the representation π equal to the direct sum of GNS-representations corresponding to all possible φ_a:

$$\pi := \bigoplus_{\varphi_a : a \in A} \pi_{\varphi_a}.$$

This representation acts in the space

$$\mathcal{H} := \bigoplus_{\varphi_a : a \in A} \mathcal{H}_{\varphi_a}.$$

Since $\|\pi_{\varphi_a}\| = \|a\|$ for any $a \in A$ we have the equality: $\|\pi(a)\| = \|a\|$, i.e. the representation π is isometric which proves the theorem. □

1.1.9 Correspondence: compact spaces $\leftrightarrow$ unital commutative Banach algebras

A continuous map $f : X \to Y$ of compact topological spaces generates the homomorphism of their algebras of continuous functions $Cf : C(Y) \to C(X)$ given by the formula: $\varphi \mapsto \varphi \circ f$. It is a unital (i.e. preserving the units) *-homomorphism with the following functorial property: if there is another continuous map $g : Y \to Z$ of compact topological spaces then $C(g \circ f) = Cf \circ Cg$.

Hence, the correspondence

$$F : X \longmapsto C(X), \quad f \longmapsto Cf$$

determines a contravariant functor from the category of compact topological spaces (with morphisms given by continuous maps) into the category of unital commutative C^*-algebras (with morphisms given by unital *-homomorphisms).

The inverse functor Φ is defined in the following way. Recall that the weak*-topology on $M(A)$ is the weakest topology for which all evaluation maps $\hat{a} : M(A) \to \mathbb{C}$, $a \in A$, are continuous. In particular, the map $f : X \to M(A)$ is continuous if and only if all maps $\hat{a} \circ f : X \to \mathbb{C}$ are continuous.

Let $\varphi : A \to B$ be a unital *-homomorphism of unital commutative C^*-algebras. Denote by $M\varphi : M(B) \to M(A)$ the map given by the formula $\mu \mapsto \mu \circ \varphi$. It is continuous since all functions of the form $\hat{a} \circ M\varphi = \widehat{\varphi(a)}$ are continuous for $a \in A$ and has the functorial property: if $\psi : B \to C$ is another unital *-homomorphism of unital commutative C^*-algebras then $M(\psi \circ \varphi) = M\varphi \circ M\psi$.

The constructed functors establish an equivalence of the above categories.

Corollary 1. *Two unital commutative C^*-algebras are isomorphic if and only if their spectra are homeomorphic.*

Corollary 2. *The group of automorphisms $\mathrm{Aut}\,A$ of a unital commutative C^*-algebra A is isomorphic to the group of homeomorphisms $\mathrm{Homeo}(M(A))$ of its spectrum.*

The constructed equivalence of categories establishes the following *dictionary* of correspondence between the topology and algebra:

topology		algebra
homeomorphism	$\longleftrightarrow$	automorphism
compactness	$\longleftrightarrow$	unitality
compactification	$\longleftrightarrow$	adding of the unit
open subset	$\longleftrightarrow$	ideal
closed subset	$\longleftrightarrow$	quotient algebra
metrizability	$\longleftrightarrow$	separability
connectedness	$\longleftrightarrow$	absence of nontrivial idempotents

We add comments only to the last two correspondences leaving the check of the others to the reader.

Proposition 1. *A compact topological space X is metrizable if and only if the algebra $C(X)$ is separable.*

Proof. If the space X is metrizable then there exists a countable family of open balls $\{U_n\}$ generating its topology. Consider the functions

$$f_n(x) := \operatorname{dist}(x, X \setminus U_n).$$

They belong to the algebra $C(X)$ and separate the points of X. Hence the subalgebra, generated by the functions f_n and constants, is dense in $C(X)$ by the Stone–Weierstrass theorem. Hence, the algebra $C(X)$ is separable.

Conversely, if $C(X)$ is separable then it contains a countable dense family of continuous functions $\{f_n\}_{n=0}^{\infty}$. We may suppose that all of them have their norms less than 1 (if one the functions f_n does not satisfy this condition we can replace it by the function $f_n/(1 + |f_n|)$). The sequence $\{f_n\}$ should separate the points of X since otherwise it could not approximate all continuous functions separating different points of X. Introduce the function

$$\operatorname{dist}(x, y) := \sum_{n=0}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n}.$$

It defines a metric on X with the balls being open subsets of X. Hence the identity map of the original space X to the same space provided with the topology, induced by the metric *dist*, is a homeomorphism which proves the metrizability of X. □

Proposition 2. *The connectedness of a compact topological space X is equivalent to the absence of nontrivial idempotents in the algebra $C(X)$.*

Proof. Recall that an *idempotent* in an algebra A is the element e such that $e^2 = e$. If the algebra $A = C(X)$ has a nontrivial idempotent e, i.e. the idempotent, differing from 1_A and zero, then such idempotent is given by a function $e \in C(X)$ which can take only two values, namely 0 and 1. Its nontriviality means that e cannot be a constant. So X decomposes into the disjoint union of two sets $\{x : \ e(x) = 1\}$ and $\{x : \ e(x) = 0\}$. Hence, X is not connected. To prove the converse statement it is sufficient to reverse the given argument. □

1.2 Vector bundles

1.2.1 Complex vector bundles

Definition 13. The *complex vector bundle* of rank r over a Hausdorff topological space M is a continuous surjective map $\pi : E \to M$ of topological spaces with fiber $E_x = \pi^{-1}(x)$ at any point $x \in M$ given by a complex vector space of dimension r such that the following *local triviality condition* holds: for any $x \in M$ there exists its open neighborhood U and a fiberwise homeomorphism

$$\varphi_U : U \times \mathbb{C}^r \longrightarrow \pi^{-1}(U)$$

which, being restricted to the fiber, is a vector space isomorphism. If it is possible to take for U in this definition the whole space M then such bundle is called *trivial*.

In analogous way one can define the C^∞-smooth vector bundles: these are the C^∞-smooth surjective maps of smooth manifolds $\pi : E \to M$ having the local triviality property.

Complex vector bundles may be defined with the help of transition functions. Let E be a complex vector bundle of rank r and $\{U_j\}$ is its *trivializing covering*, in other words, $\{U_j\}$ is an open covering of the space M together with homeomorphisms

$$\varphi_j \equiv \varphi_{U_j} : U_j \times \mathbb{C}^r \longrightarrow \pi^{-1}(U_j)$$

having the properties listed in Definition 13. Then the functions $\varphi_{ij} := \varphi_j^{-1} \circ \varphi_i$, defined on the intersections $U_{ij} := U_i \cap U_j$, are called the *transition functions* of the bundle E. They take values in the group $\mathrm{GL}(r, \mathbb{C})$ of nondegenerate linear maps $\mathbb{C}^r \to \mathbb{C}^r$ and satisfy the following *cocycle condition*:

$$\varphi_{ii} = \mathrm{id}, \quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \text{ on } U_{ijk} := U_i \cap U_j \cap U_k.$$

Denote by $\Gamma(U, E)$ the set of *sections* of the bundle E over a subset $U \subset M$, i.e. continuous maps $s : U \to E$ such that $\pi \circ s = \mathrm{id}$.

Proposition 3. *Let E be a complex vector bundle of rank r over a compact topological manifold M. Then there exists another complex vector bundle $E' \to M$ such that for some n we shall have the following isomorphism of vector bundles*

$$E \oplus E' \cong M \times \mathbb{C}^n.$$

Remark 2. In the real case if E coincides with the tangent bundle of the manifold M embedded into a vector space, one can take for E' the normal bundle of this manifold.

Proof. Using the compactness of M, we can choose a finite trivializing covering $\{U_j\}_{j=1}^m$ of M. For every set U_j from this covering there exists a collection of r linearly independent sections $s_{j1}, \ldots, s_{jr} : U_j \to E$ of the bundle E. Denote by $\{\psi_j\}_{j=1}^m$ a continuous decomposition of unity subordinate to the covering $\{U_j\}$, consisting of continuous functions ψ_j with compact support in U_j, for which $\sum \psi_j \equiv 1$. Introduce the maps

$$\sigma_{jk} : M \to E, \text{ equal to } \begin{cases} \psi_j s_{jk} & \text{on } U_j \\ 0 & \text{outside } U_j. \end{cases}$$

The vectors $\sigma_{j1}(x), \ldots, \sigma_{jr}(x)$ generate the fiber E_x for any $x \in M$.

Set $n := mr$ and consider the map $\beta : M \times \mathbb{C}^n \to E$ defined by the formula

$$\beta(x, t) = \sum_{j,k} t_{jk} \sigma_{jk}(x).$$

It determines a surjective bundle morphism (i.e. a fiber-linear map) which can be included into the following exact sequence of morphisms

$$0 \longrightarrow E' := \operatorname{Ker} \beta \xrightarrow{\alpha} M \times \mathbb{C}^n \xrightarrow{\beta} E \longrightarrow 0.$$

We show that this sequence *splits*, i.e. there exists a right inverse morphism to morphism β. This will imply that $E \oplus E' \cong M \times \mathbb{C}^n$.

Indeed, at any point $x \in M$ the exact sequence of linear maps of vector space

$$0 \longrightarrow E'_x \xrightarrow{\alpha_x} \mathbb{C}^n \xrightarrow{\beta_x} E_x \longrightarrow 0$$

splits since $\dim \operatorname{Ker} \beta_x + \dim \operatorname{Im} \beta_x = n$. In some neighborhood U_x of the point x the morhism β is given by a matrix function $b : U_x \to \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ having at the point x the rank equal to $\operatorname{rk} b(x) = r$. By restricting, if necessary, the neighborhood U_x to a neighborhood $V_x \subset U_x$ we can assert that $\operatorname{rk} b(y) = r$ for any $y \in V_x$. The exact sequence

$$0 \longrightarrow E' \longrightarrow M \times \mathbb{C}^n \longrightarrow E \longrightarrow 0$$

over this neighborhood V_x still splits. We choose now from the collection $\{V_x\}_{x \in M}$ a finite subcovering $\{V_j\}$ of M and denote by γ_j the morphism which is the right inverse to β over the neighborhood V_j. For a continuous decomposition of unity $\{\chi_j\}$ subordinate to the covering $\{V_j\}$ we set

$$\gamma := \sum_j \chi_j \gamma_j.$$

Then γ is the right inverse to β over M. □

Remark 3. The proved proposition remains true in the case when M is a paracompact manifold.

1.2.2 Functor Γ

The space of sections $\Gamma(M, E) \equiv \Gamma(E)$ of a vector bundle $\pi : E \to M$ is a module over the commutative Banach algebra $C(M)$ which we consider as a right module by setting

$$(sa)(x) := s(x)a(x) \quad \text{for } s \in \Gamma(E), a \in C(M).$$

It is a covariant functor from the category of vector bundles over M into the category of right modules over the algebra $C(M)$. Indeed, we can associate with any bundle morphism $\tau : E \to E'$ a homomorphism of $C(M)$-modules

$$\Gamma\tau : \Gamma(E) \longrightarrow \Gamma(E')$$

acting by the formula: $(\Gamma\tau)s = \tau \circ s$. This homomorphism is linear, i.e. $(\Gamma\tau)(sa) = (\Gamma\tau)(s)a$ for $a \in C(M)$ since the maps $\tau_x : E_x \to E'_x$ are linear for $x \in M$. Moreover, the functor Γ transforms the operations of duality, direct sum and tensor product for the bundles into the analogous operations for $C(M)$-modules.

Apart from that, this functor has the following important properties:

1. Γ is *faithful* which means that the equality $\Gamma f = \Gamma g$ for two bundle morphism $f, g : E \to E'$ implies the equality $f = g$.

2. Γ is *full* which means that the map

$$\tau \longmapsto \Gamma\tau : \operatorname{Hom}(E, E') \longrightarrow \operatorname{Hom}_{C(M)}(\Gamma(E), \Gamma(E'))$$

is surjective.

3. Γ preserves the short exact sequences and transforms the splitting short exact sequences again into the splitting short exact sequences.

1.2.3 Projective modules

In this section $\mathcal{A}$ denotes a unital ring.

Definition 14. A right $\mathcal{A}$-module $\mathcal{E}$ over a unital ring $\mathcal{A}$ is called *free* if it has an $\mathcal{A}$-*basis*, i.e. a set of generators T such that every relation of the form $t_1 a_1 + \ldots + t_r a_r = 0$ with $t_j \in T$, $a_j \in \mathcal{A}$, implies that $a_1 = \ldots = a_r = 0$. A module $\mathcal{E}$ is called *finitely generated* if it has a finite set of generators or a finite $\mathcal{A}$-basis.

Example 3. The standard free $\mathcal{A}$-module of rank r has the form $\mathcal{A}^r = \underbrace{\mathcal{A} \oplus \ldots \oplus \mathcal{A}}_{r}$ and consists of vector-columns with entries from $\mathcal{A}$. It has the standard basis consisting of elements $e_j = {}^t(0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 at jth place). It may be also realized as a module ${}^r\mathcal{A} = \underbrace{\mathcal{A} \oplus \ldots \oplus \mathcal{A}}_{r}$ consisting of vector-rows with entries from $\mathcal{A}$. Any finitely generated free $\mathcal{A}$-module is isomorphic to $\mathcal{A}^r$ and ${}^r\mathcal{A}$ for some r.

Definition 15. A right $\mathcal{A}$-module $\mathcal{P}$ is called *projective* if it has the following universal property: for any surjective $\mathcal{A}$-linear map of right $\mathcal{A}$-modules $f : \mathcal{E} \to \mathcal{G}$ and any $\mathcal{A}$-linear map of right $\mathcal{A}$-modules $\varphi : \mathcal{P} \to \mathcal{G}$ there exists an $\mathcal{A}$-linear map $\psi : \mathcal{P} \to \mathcal{E}$ such that the following diagram

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{f} & \mathcal{G} & \longrightarrow & 0 \\ {\scriptstyle\psi}\uparrow & \nearrow_{\varphi} & & & \\ \mathcal{P} & & & & \end{array}$$

is commutative.

Properties of projective modules:

1. Any free right $\mathcal{A}$-module is projective.

2. The direct sum of right $\mathcal{A}$-modules is projective $\Longleftrightarrow$ every summand in this sum is projective.

3. A right $\mathcal{A}$-module $\mathcal{P}$ is projective $\Longleftrightarrow$ any short exact sequence of $\mathcal{A}$-linear maps of right $\mathcal{A}$-modules of the form

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow \mathcal{P} \longrightarrow 0 \tag{1.4}$$

   splits.

4. A right $\mathcal{A}$-module $\mathcal{P}$ is projective $\Longleftrightarrow$ $\mathcal{P}$ is a direct summand in a free $\mathcal{A}$-module.

5. A right $\mathcal{A}$-module $\mathcal{P}$ is projective $\Longleftrightarrow$ it has the form $\mathcal{P} = e\mathcal{F}$ where $\mathcal{F}$ is a right free $\mathcal{A}$-module and e is an idempotent in the algebra $\mathrm{End}_{\mathcal{A}}\mathcal{F}$ of $\mathcal{A}$-linear endomorphisms of $\mathcal{F}$.

6. If a projective $\mathcal{A}$-module is *finitely generated*, , i.e. it has a finite system of generators, then the free module in properties 4 and 5 can be also chosen finitely generated.

Proofs of properties of projective modules

Proof of property (1). Let $\mathcal{F}$ be a free right $\mathcal{A}$-module with a system of generators $\{t_\alpha\}_{\alpha\in I}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a $\mathcal{A}$-linear map of right $\mathcal{A}$-modules and $f : \mathcal{E} \to \mathcal{G}$ is a surjective $\mathcal{A}$-linear map. Since this map is surjective we can find for any generator t_α of the module $\mathcal{F}$ an element $s_\alpha \in \mathcal{E}$ such that $f(s_\alpha) = \varphi(t_\alpha)$. Define the map $\psi : \mathcal{F} \to \mathcal{E}$ by setting it equal to $\psi(t_\alpha) = s_\alpha$ on generators and extending further on by $\mathcal{A}$-linearity. The constructed diagram

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{f} & \mathcal{G} & \longrightarrow & 0 \\ {\scriptstyle\psi}\uparrow & \nearrow{\scriptstyle\varphi} & & & \\ \mathcal{F} & & & & \end{array}$$

is commutative since on generators we have: $f(\psi(t_\alpha)) = f(s_\alpha) = \varphi(t_\alpha)$ and by $\mathcal{A}$-linearity the equality $f(\psi(t)) = \varphi(t)$ holds for all $t \in \mathcal{F}$. □

Proof of property (2). Suppose that the module $\mathcal{P} = \bigoplus_{\alpha\in I} \mathcal{P}_\alpha$, which is the direct sum of right $\mathcal{A}$-modules, is projective. We show that in this case any $\mathcal{A}$-module $\mathcal{P}_\alpha$ is also projective. Suppose that we are given an $\mathcal{A}$-linear map of right $\mathcal{A}$-modules $\varphi_\alpha : \mathcal{P}_\alpha \to \mathcal{G}$ and surjective $\mathcal{A}$-linear map of right $\mathcal{A}$-modules $f : \mathcal{E} \to \mathcal{G}$. The map φ_α may be extended to an $\mathcal{A}$-linear map $\varphi : \mathcal{P} \to \mathcal{G}$ by setting it equal to zero on all summands $\mathcal{P}_\beta$ with $\beta \neq \alpha$. Since the module $\mathcal{P}$ is projective it should exist an $\mathcal{A}$-linear map $\psi : \mathcal{P} \to \mathcal{E}$ such that $f \circ \psi = \varphi$. By restricting this map to $\mathcal{P}_\alpha$ we obtain an $\mathcal{A}$-linear map $\psi_\alpha : \mathcal{P}_\alpha \to \mathcal{E}$ such that $f \circ \psi_\alpha = \varphi_\alpha$. This proves that the module $\mathcal{P}_\alpha$ is projective.

Conversely, suppose that all $\mathcal{A}$-modules $\mathcal{P}_\alpha$ are projective. Let $\varphi : \mathcal{P} \to \mathcal{G}$ be an $\mathcal{A}$-linear map of right $\mathcal{A}$-modules and $f : \mathcal{E} \to \mathcal{G}$ is a surjective $\mathcal{A}$-linear map of right $\mathcal{A}$-modules. For every $\alpha \in I$ the restriction $\varphi_\alpha := \varphi|\mathcal{P}_\alpha$ is an $\mathcal{A}$-linear map $\mathcal{P}_\alpha \to \mathcal{G}$ so due to the projectivity of the $\mathcal{A}$-module $\mathcal{P}_\alpha$ there exists an $\mathcal{A}$-linear map $\psi_\alpha : \mathcal{P}_\alpha \to \mathcal{E}$ such that $f \circ \psi_\alpha = \varphi_\alpha$. Then the direct sum $\psi := \bigoplus_{\alpha\in I} \psi_\alpha$ of the maps ψ_α will give an $\mathcal{A}$-linear map $\psi : \mathcal{P} \to \mathcal{E}$ with the property: $f \circ \psi = \varphi$ which implies that the module $\mathcal{P}$ is projective. □

Proof of properties (3) and (4). We prove first that if a right $\mathcal{A}$-module $\mathcal{P}$ is projective then every short exact sequence of the form

$$0 \longrightarrow \mathcal{E} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{P} \longrightarrow 0,$$

where $\mathcal{G}$ is a right $\mathcal{A}$-module, splits. Indeed, due to the projectivity of the module $\mathcal{P}$ there exists an $\mathcal{A}$-linear map $\psi : \mathcal{P} \to \mathcal{G}$ for which the diagram

$$\begin{array}{ccccc} \mathcal{G} & \xrightarrow{g} & \mathcal{P} & \longrightarrow & 0 \\ {\scriptstyle\psi}\uparrow & \nearrow{\scriptstyle \mathrm{id}_{\mathcal{P}}} & & & \\ \mathcal{P} & & & & \end{array}$$

is commutative, i.e. $g \circ \psi = \mathrm{id}_{\mathcal{P}}$. Hence, the map ψ is the right inverse of g, i.e. the considered exact sequence splits.

We prove now that for any projective right $\mathcal{A}$-module $\mathcal{P}$ there exists an exact sequence of the form (1.4) in which $\mathcal{G} \equiv \mathcal{F}$ is a free right $\mathcal{A}$-module. Choose in $\mathcal{P}$ some system of generators $\{t_\alpha\}_{\alpha \in I}$ and consider the free right $\mathcal{A}$-module $\mathcal{F}$ with an $\mathcal{A}$-basis $\{s_\alpha\}_{\alpha \in I}$ parameterized by the same set of indices I. In other words, the module $\mathcal{F}$ consists of all finite linear combinations $\sum_{k=1}^{n} s_{\alpha_k} a_k$ where $a_k \in \mathcal{A}$, $k = 1, \ldots, n$. Define the map $g : \mathcal{F} \to \mathcal{P}$ on generators by the formula: $g(s_\alpha) = t_\alpha$ and extend it to the whole module $\mathcal{F}$ by $\mathcal{A}$-linearity. Setting $\mathcal{E} := \operatorname{Ker} g$ and denoting by f the embedding $\mathcal{E} \hookrightarrow \mathcal{F}$, we shall obtain the exact short sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{P} \longrightarrow 0. \tag{1.5}$$

If for a right $\mathcal{A}$-module $\mathcal{P}$ any exact sequence of the form (1.4) splits then this property holds also for the above sequence (1.5) which implies that the module $\mathcal{P}$ is a direct summand in the free $\mathcal{A}$-module $\mathcal{F}$. By property (1) the free module $\mathcal{F}$ is projective and by property (2) the $\mathcal{A}$-module $\mathcal{P}$, which is a direct summand in the module $\mathcal{F}$, should be also projective.

To finish the proof of the property (4) it remains to note that if an $\mathcal{A}$-module $\mathcal{P}$ is a direct summand in a free $\mathcal{A}$-module $\mathcal{F}$, which is projective by property (1), then it is projective by property (2). □

Proof of property (5). If a right $\mathcal{A}$-module $\mathcal{P}$ is projective then by property (4) it is a direct summand in a free right $\mathcal{A}$-module $\mathcal{F}$ and we can take for the desired idempotent the projector $p : \mathcal{F} \to \mathcal{P}$.

Conversely, if $e \in \mathrm{End}_{\mathcal{A}}\mathcal{F}$ is an idempotent, i.e. $e^2 = e$, acting in a free right $\mathcal{A}$-module $\mathcal{F}$, and the right $\mathcal{A}$-module $\mathcal{P}$ has the form $\mathcal{P} = e\mathcal{F}$, then

$$\mathcal{F} = e\mathcal{F} \oplus (\mathrm{id}_{\mathcal{F}} - e)\mathcal{F}$$

where the map $\mathrm{id}_{\mathcal{F}} - e$ is also an idempotent. So the module $\mathcal{P} = e\mathcal{F}$ is a direct summand in the free $\mathcal{A}$-module $\mathcal{F}$, hence a projective module. □

Proof of property (6). If $\mathcal{P}$ is a finitely generated right $\mathcal{A}$-module with a system of generators $\{t_\alpha\}_{\alpha=1}^{r}$ then in the proof of property (4) the $\mathcal{A}$-basis $\{s_\alpha\}_{\alpha=1}^{r}$ of the free $\mathcal{A}$-module $\mathcal{F}$ will be also finite. Hence, $\mathcal{P}$ is a direct summand in the finite-dimensional free $\mathcal{A}$-module $\mathcal{F}$ so coincides with the image of an idempotent e, acting in $\mathcal{F}$. □

1.2.4 Serre–Swan theorem

Proposition 4. *Let M be a compact manifold and $E \to M$ is a complex vector bundle of rank r over it. Then the $C(M)$-module $\Gamma(E)$ is finitely generated and projective.*

Proof. According to Proposition 3 there exists a vector bundle $E' \to M$ such that $E \oplus E' \cong M \times \mathbb{C}^n$. Since

$$\Gamma(E) \oplus \Gamma(E') = \Gamma(M \times \mathbb{C}^n) = C(M)^n$$

the module $\Gamma(E)$ is a direct summand in the free module $C(M)^n$. It implies also that $\Gamma(E)$ is finitely generated. □

Theorem 4 (Serre–Swan)**.** *The functor* Γ *establishes an equivalence between the category of vector bundles over the compact manifold* M *and the category of finitely generated projective modules over the algebra* $C(M)$*.*

Proof. Due to Proposition 4 it remans to prove that any finitely generated projective $C(M)$-module $\mathcal{E}$ coincides with $\Gamma(M,E)$ for some vector bundle $\pi : E \to M$. By property (5) of projective modules (cf. Sec. 1.2.3) the module $\mathcal{E}$ has the form $\mathcal{E} = eC(M)^n$ for some idempotent $e \in \mathrm{End}_{C(M)}(C(M)^n) \cong \mathrm{Mat}_n(C(M))$. The exact sequence

$$0 \longrightarrow \mathrm{Ker}\, e \longrightarrow C(M)^n \longrightarrow \mathcal{E} \longrightarrow 0$$

splits by property (3) of projective modules (cf. Sec. 1.2.3). Since the functor Γ is full it follows that the endomorphism $e \in \mathrm{Mat}_n(C(M))$ is generated by some bundle morphism $\tau : M \times \mathbb{C}^n \to M \times \mathbb{C}^n$ so that $eC(M)^n$ coincides with $\Gamma(M,E)$ where $E := \mathrm{Im}\,\tau$.

It remains to show that E is a subbundle of $M \times \mathbb{C}^n$. For that it is sufficient to prove that $\mathrm{rk}\,\tau_x$ is a locally constant function of $x \in X$. Note that the rank of idempotent τ_x, as any other idempotent in $\mathrm{Mat}_n(C(M))$, is upper semicontinuous. On the other hand, $\mathrm{id} - \tau$ is also an idempotent in $\mathrm{Mat}_n(C(M))$ and $\mathrm{rk}(\mathrm{id}_x - \tau_x) = n - \mathrm{rk}\,\tau_x$ for any $x \in M$. It follows that the map $x \mapsto \mathrm{rk}\,\tau_x$ is not only upper but also lower semicontinuous, so it is continuous, hence, locally constant. □

Remark 4. The given proof admits the following interpretation. The idempotent $e \in \mathrm{Mat}_n(C(M))$ may be considered as a continuous map from M into the space of matrix idempotents. Denote by ε_x the evaluation map at the point $x \in M$. As it was shown in the proof of the theorem, the map $\varepsilon(e) : x \mapsto \varepsilon_x(e)$ has constant rank $m = \mathrm{rk}\,\varepsilon_x(e) \leq n$, hence it generates a map $M \to G_m(\mathbb{C}^n)$ into the Grassmann manifold $G_m(\mathbb{C}^n)$. In other words, the space $E_x = \varepsilon_x(e)\mathbb{C}^n$ lies in the fibre of the tautological bundle $T \to G_m(\mathbb{C}^n)$ over the point x. The desired bundle $E \to M$, corresponding to the projective module $\mathcal{E}$, coincides with the inverse image of the tautological bundle T under the map $\varepsilon(e)$:

$$\begin{array}{ccc} E = \varepsilon(e)^*(T) & \longrightarrow & T \\ \downarrow & & \downarrow \\ M & \xrightarrow[\varepsilon(e)]{} & G_m(\mathbb{C}^n) \end{array}$$

Remark 5. The C^∞-smooth version of Serre–Swan theorem is also true. It asserts that the category of C^∞-smooth vector bundles over a C^∞-smooth compact manifold M is equivalent to the category of finitely generated projective modules over the algebra $C^\infty(M)$. The equivalence of these categories is established by the functor Γ^∞ associating with a smooth bundle $E \to M$ the module $\Gamma^\infty(M,E)$ of its C^∞-smooth sections.

1.3 Functional analysis over C^*-algebras

1.3.1 C^*-modules

Definition 16. The (right) C^*-*premodule* over a C^*-algebra A is a complex vector space $\mathcal{E}$ which is simultaneously a right A-module provided with a sesquilinear pairing $(\cdot\,,\cdot):\mathcal{E}\times\mathcal{E}\to A$ having the following properties:

$$\begin{aligned}&(1)\quad (r,s+t)=(r,s)+(r,t);\\&(2)\quad (r,sa)=(r,s)\,a;\\&(3)\quad (r,s)=(s,r)^*;\\&(4)\quad (s,s)>0 \text{ for } s\neq 0,\end{aligned}$$

satisfied for any $r,s,t\in\mathcal{E}$, $a\in A$.

It follows that the pairing $(\cdot\,,\cdot)$ is A-linear in second variable, antilinear in the first variable and positively defined. In particular, $(ra,s)=a^*(r,s)$ for $a\in A$, $r,s\in\mathcal{E}$.

Using the pairing $(\cdot\,,\cdot)$ we can introduce on the C^*-premodule $\mathcal{E}$ the *norm* by setting

$$\|s\|_{\mathcal{E}}:=\sqrt{\|(s,s)\|}$$

for $s\in\mathcal{E}$. Here, $\|\cdot\|$ denotes the norm in the C^*-algebra A. (We shall omit in the sequel the lower index $\mathcal{E}$ in the notation of the norm $\|\cdot\|_{\mathcal{E}}$ on the C^*-premodule $\mathcal{E}$ when it is clear which norm is considered.)

The following lemma is an analogue of the Cauchy inequality for the introduced norm.

Lemma 4. *Let $\mathcal{E}$ be a C^*-premodule over a C^*-algebra A. Then for any $r,s\in\mathcal{E}$ the following inequality holds*

$$\|(r,s)\|\le\sqrt{\|(r,r)\|}\sqrt{\|(s,s)\|}.$$

Proof. We shall give the proof of this inequality only in the case of a unital algebra A leaving the general case as an exercise. For an arbitrary element $a\in A$ we have the following relationn

$$0\le(ra+s,ra+s)=a^*(r,r)a+a^*(r,s)+(s,r)a+(s,s). \tag{1.6}$$

Since (r,r) and (s,s) are positive elements of A they satisfy the estimates: $a^*(r,r)a\le\|(r,r)\|a^*a$ and $(s,s)\le\|(s,s)\|\cdot 1_A$. So the inequality (1.6) implies that

$$0\le(ra+s,ra+s)=\|(r,r)\|a^*a+a^*(r,s)+(s,r)a+\|(s,s)\|\cdot 1_A.$$

Setting $a=-\frac{(r,s)}{(r,r)}$ in the last relation we get

$$0\le-\frac{(s,r)(s,r)^*}{\|(r,r)\|}+\|(s,s)\|\cdot 1_A.$$

Hence,

$$0\le(s,r)(s,r)^*\le\|(r,r)\|\cdot\|(s,s)\|\cdot 1_A$$

which implies the desired inequality. □

Using the proved lemma it is easy to check that $\|s\|_{\mathcal{E}}=\sqrt{\|(s,s)\|}$ is indeed a norm on $\mathcal{E}$, in particular, it satisfies the triangle inequality.

Definition 17. A C^*-premodule $\mathcal{E}$ is called the *C^*-module* if it is complete with respect to the norm $\|s\|_{\mathcal{E}}$.

In particular, the completion of an arbitrary C^*-premodule with respect to the norm $\|s\|_{\mathcal{E}}$ is a C^*-module.

Examples of C^*-modules

1. A complex Hilbert space is a C^*-module over the algebra $\mathbb{C}$ with the pairing given by the inner product.

2. Any C^*-algebra A is a C^*-module over itself with the pairing given by the formula: $(a, b) := a^*b$.

3. The free A-module A^n, consisting of vector-columns with entries from A, is a C^*-module over A with the pairing of elements $a = {}^t(a_1, \dots, a_n)$ and $b = {}^t(b_1, \dots, b_n)$ given by the formula: $(a, b) := (a_1^*, \dots, a_n^*)\ {}^t(b_1, \dots, b_n) = a_1^*b_1 + \dots + a_n^*b_n$.

4. The free A-module ${}^nA$, consisting of vector-rows with entries from A, is a C^*-module over the algebra $\mathrm{Mat}_n(A)$ with the pairing of elements $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ given by the formula: $(a, b) := {}^t(a_1^*, \dots, a_n^*)(b_1, \dots, b_n) = (a_i^*b_j)_{i,j=1}^n$.

An important example of a C^*-module over the C^*-algebra A is given by the tensor product $\mathcal{H} \otimes A$ where $\mathcal{H}$ is a Hilbert space. First of all we should precisely define the meaning of the tensor product used in this formula. Recall that the *algebraic tensor product* $\mathcal{H} \odot A$ of a Hilbert space $\mathcal{H}$ and a C^*-algebra A is the right A-module, consisting of finite sums of pure tensors of the form $\sum_{k=1}^n \xi_k \otimes a_k$ with $\xi_k \in \mathcal{H}$, $a_k \in A$, provided with an A-valued pairing given on pure tensors by the formula

$$(\xi \otimes a, \eta \otimes b) = (\xi, \eta)a^*b$$

where $a, b \in A$, $\xi, \eta \in \mathcal{H}$. It can be shown that this pairing has all properties listed in Definition 16, in particular, it is positively defined. Hence, $\mathcal{H} \odot A$ is a C^*-premodule over A. Its completion with respect to the norm, induced by the introduced pairing, is a C^*-module over A which is denoted by $\mathcal{H} \otimes A$ and called the *tensor product of a Hilbert space $\mathcal{H}$ and the algebra A* .

We shall use also the C^*-module $\mathcal{H}_A \equiv \ell^2_A$ over A consisting of sequences $a = \{a_k\}$ of elements from A for which the series

$$\sum_{k=1}^{\infty} a_k^* a_k$$

converges in A. It can be provided with the sesquilinear pairing of the form

$$(a, b) := \sum_{k=1}^{\infty} a_k^* b_k.$$

As a first example of linear operators, acting in $\mathcal{H}_A$, consider the operators P_n of the form

$$P_n(a_1, \dots, a_n, a_{n+1}, \dots) = (a_1, \dots, a_n, 0, \dots)$$

which are, evidently, *projectors* in $\mathcal{H}_A$, i.e. $P_n^2 = P_n$ and $P_n^* = P_n$. Also $\|\xi - P_n\xi\| \to 0$ for $n \to \infty$ for any $\xi \in \mathcal{H}$.

We shall return to the study of linear operators, acting in C^*-modules, in Sec. 1.3.3. Consider now in more detail the notion of the tensor product we have met already in this subsection.

1.3.2 Tensor products

Functor E_φ

Let $\mathcal{A}$, $\mathcal{B}$ be unital rings (i.e. rings with units) and $\varphi : \mathcal{A} \to \mathcal{B}$ is a unital (i.e. sending $1_\mathcal{A}$ to $1_\mathcal{B}$) homomorphism. We can associate with such homomorphism φ the functor E_φ from the category of right $\mathcal{A}$-modules into the category of right $\mathcal{B}$-modules defined in the following way.

Let $\mathcal{E}$ be a right $\mathcal{A}$-module. Then the ring $\mathcal{B}$ may be provided with the structure of left $\mathcal{A}$-module by setting: $a \cdot b := \varphi(a)b$. This allows us to define the tensor product

$$E_\varphi(\mathcal{E}) = \mathcal{E} \otimes_\mathcal{A} \mathcal{B}$$

which will be called the *tensor product* of the right $\mathcal{A}$-module $\mathcal{E}$ and the ring $\mathcal{B}$ *with respect to the homomorphism* φ.

Namely, $\mathcal{E} \otimes_\mathcal{A} \mathcal{B}$ is an Abelian group with elements being finite sums of the form $\sum_{k=1}^n s_k \otimes b_k$, $s_k \in \mathcal{E}$, $b_k \in \mathcal{B}$, provided with the unique relation

$$(sa) \otimes b = s \otimes \varphi(a)b \quad \text{for any } a \in \mathcal{A}.$$

Introduce the right action of $\mathcal{B}$ on $\mathcal{E} \otimes_\mathcal{A} \mathcal{B}$ by setting: $(s \otimes b)c := s \otimes (bc)$ for all $s \in \mathcal{E}$, $b, c \in \mathcal{B}$.

If $\mathcal{F}$ is another right $\mathcal{A}$-module and $\tau : \mathcal{E} \to \mathcal{F}$ is an $\mathcal{A}$-linear map then $E_\varphi(\tau) := \tau \otimes \mathrm{id}_\mathcal{B}$ is a $\mathcal{B}$-linear map from $\mathcal{E} \otimes_\mathcal{A} \mathcal{B}$ into $\mathcal{F} \otimes_\mathcal{A} \mathcal{B}$. Hence, E_φ does define a functor from the category of right $\mathcal{A}$-modules into the category of right $\mathcal{B}$-modules.

Tensor product of C^*-algebras

Recall first the definition of the tensor product of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Consider the algebraic tensor product $\mathcal{H}_1 \odot \mathcal{H}_2$ with elements given by the finite sums of pure tensors of the form $\sum_{k=1}^n \xi_k \odot \eta_k$. We can introduce on the space $\mathcal{H}_1 \odot \mathcal{H}_2$ the structure of a pre-Hilbert space with the pairing given on pure tensors by the formula

$$(\xi_1 \odot \eta_1, \xi_2 \odot \eta_2) := (\xi_1, \xi_2)(\eta_1, \eta_2)$$

where $\xi_1, \xi_2 \in \mathcal{H}_1$, $\eta_1, \eta_2 \in \mathcal{H}_2$.

This pairing is positively defined and the *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to the norm generated by this pairing. Note that the introduced norm has the *cross property* which may be written for pure tensors $\xi \otimes \eta \in \mathcal{H}_1 \otimes \mathcal{H}_2$ in the form

$$\|\xi \otimes \eta\| = \|\xi\| \cdot \|\eta\|.$$

The norms, having this property, will be called the *cross-norms*.

In the case of Banach spaces E_1,E_2 the algebraic tensor product $E_1 \odot E_2$ may have several different norms with the cross property leading to different completions.

We switch now to the case of C^*-algebras A, B and consider the cross-norms on the algebraic tensor product $A \odot B$ which have also the C^*-property, i.e. $\|z^*z\| = \|z\|^2$ for all $z \in A \odot B$.

We call a C^*-algebra A *nuclear* if the algebraic tensor product $A \odot B$ with any other C^*-algebra B has a unique C^*-cross-norm. The completion of this product with respect to this norm is called the *tensor product* $A \otimes B$ of C^*-algebras A and B.

The examples of nuclear C^*-algebras are given by the finite-dimensional C^*-algebras $\mathrm{Mat}_n(\mathbb{C})$ for which $\mathrm{Mat}_n(\mathbb{C}) \otimes B \cong \mathrm{Mat}_n(B)$ and commutative C^*-algebras $C_0(Y)$ for which $C_0(Y) \otimes B \cong C_0(Y, B)$. The algebra $\mathcal{K}(\mathcal{H})$ of compact operators in a Hilbert space $\mathcal{H}$ yields one more important example of nuclear C^*-algebras. However, the C^*-algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators in an infinite-dimensional Hilbert space $\mathcal{H}$ is already not nuclear.

Tensor product of C^*-modules

Consider first the construction of the tensor product of a right C^*-module $\mathcal{E}$ over a C^*-algebra A with another C^*-algebra B extending the construction of the above functor E_φ to the case of C^*-modules and C^*-algebras.

Suppose that $\varphi : A \to B$ is a morphism of C^*-algebras. We want to define the tensor product $E_\varphi(\mathcal{E}) = \mathcal{E} \otimes_A B$.

Consider the algebraic tensor product $\mathcal{E} \odot B$ over $\mathbb{C}$ and introduce the B-valued pairing on it given on pure tensors $s \odot b$ by the formula

$$(s \odot b, t \odot c) := b^* \varphi((s,t))c \tag{1.7}$$

where $s, t \in \mathcal{E}$, $b, c \in B$.

However, the introduced pairing has the kernel and in order to introduce a norm on $\mathcal{E} \odot B$ it is necessary first to factor this kernel out. It can be shown (cf. [3], p.73) that the kernel $N := \{z \in \mathcal{E} \odot B : (z,z) = 0\}$ is generated by the elements of the form $sa \odot b - s \odot \varphi(a)b$. Restricting to the quotient $(\mathcal{E} \odot B)/N$, we obtain a positive definite pairing on it which is defined on pure tensors $s \otimes t \in (\mathcal{E} \odot B)/N$ by the same formula

$$(s \otimes b, t \otimes c) = b^* \varphi((s,t))c.$$

It follows that the quotient $(\mathcal{E} \odot B)/N$ is a C^*-premodule over the algebra B and its completion with respect to the norm, generated by the introduced pairing, is a C^*-module. We denote it by $E_\varphi(\mathcal{E}) = \mathcal{E} \otimes_A B$ and call the *tensor product of the C^*-module $\mathcal{E}$ and the C^*-algebra B.*

The tensor product of C^*-modules is defined following the same scheme. Suppose that we have two right C^*-modules $\mathcal{E}$ and $\mathcal{F}$ over the C^*-algebras A and B respectively. Suppose moreover that we are given a morphism $\rho : A \to \mathrm{End}_B \mathcal{F}$.

Provide the algebraic tensor product $\mathcal{E} \odot \mathcal{F}$, being a right B-module, with the pairing given on pure tensors by the formula

$$(s_1 \odot t_1, s_2 \odot t_2) = (t_1, \rho\left((s_1, s_2)\right) t_2) = (\rho\left((s_2, s_1)\right) t_1, t_2) \tag{1.8}$$

where $s_1, s_2 \in \mathcal{E}$, $t_1, t_2 \in \mathcal{F}$. Consider again the B-submodule in $\mathcal{E} \odot \mathcal{F}$ consisting of the elements $z \in \mathcal{E} \odot \mathcal{F}$ such that $(z,z) = 0$. This submodule is generated, as before, by the elements of the form $sa \odot - s \odot \rho(a)t$.

The pairing (1.8) defines a positive definite inner product on the quotient $(\mathcal{E} \odot \mathcal{F})/N$. The completion of $(\mathcal{E} \odot \mathcal{F})/N$ with respect to the norm, generated by this inner product, is a C^*-module which is denoted by $\mathcal{E} \otimes_\rho \mathcal{F} = \mathcal{E} \otimes_A \mathcal{F}$ and called the *tensor product of the C^*-module $\mathcal{E}$ and C^*-module $\mathcal{F}$.*

1.3.3 Operators of A-finite rank and A-compact operators

We start the study of linear operators acting in C^*-modules from the simplest operators of rank 1. These operators, acting from a C^*-module $\mathcal{E}$ to a C^*-module $\mathcal{F}$, are called the *ketbra-operators* and denoted by $|r\rangle\langle s|$ where $s \in \mathcal{E}$, $r \in \mathcal{F}$. The operator $|r\rangle\langle s|$ is defined by the formula

$$|r\rangle\langle s| : \ t \longmapsto r(s,t)$$

where $t \in \mathcal{E}$. It is an A-linear operator since $r(s, ta) = r(s,t)a$ for $a \in A$. Its adjoint is again the ketbra-operator given by the formula $(|r\rangle\langle s|)^* = |s\rangle\langle r|$.

In the case when $\mathcal{E} = \mathcal{F}$ the composition of two ketbra-operators

$$|r\rangle\langle s| \circ |t\rangle\langle u| = |r\rangle(s,t)\langle u| = |r\rangle\langle u(t,s)|$$

is again a ketbra-operator so that the finite sums of ketbra-operators form an algebra of operators acting in $\mathcal{E}$. This algebra is a two-sided ideal $\mathrm{Fin}_A\mathcal{E}$ in the algebra $\mathrm{End}_A\mathcal{E}$. Its completion with respect to the operator norm is denoted by $\mathcal{K}_A(\mathcal{E})$.

More generally, let us denote by $\mathrm{Fin}_A(\mathcal{E}, \mathcal{F})$ the space of ketbra-operators of the form

$$\sum_{k=1}^{n} |r_k\rangle\langle s_k|$$

called otherwise the operators of *A-finite rank*. The completion of $\mathrm{Fin}_A(\mathcal{E}, \mathcal{F})$ with respect to the operator norm is denoted by $\mathcal{K}_A(\mathcal{E}, \mathcal{F})$ and its elements are called the *A-compact operators*.

Note that an A-compact operator need not be compact in the usual sense.

Example 4. For any unital C^*-algebra A we have an isomorphism

$$\mathcal{K}_A(A) \cong A.$$

Indeed, the map $T \mapsto T(1_A)$ determines a bijection of the algebra of bounded A-linear operators acting in A onto the algebra A. Moreover, any operator of the form $a \mapsto b^*a$, which coincides in fact with the ketbra-operator $|1_A\rangle\langle b|$, is adjointable and has finite A-rank. Hence $\mathcal{K}_A(A) \cong A$.

In analogous way it can be shown that

$$\mathcal{K}_A(A^n) \cong \mathrm{Mat}_n(A).$$

Later we show that for the algebra $A = C(M)$, where M is a compact manifold, we have an isomorphism

$$\mathcal{K}_A(\Gamma(M, E)) \cong \Gamma(M, \mathrm{End}\, E)$$

for any Hermitian vector bundle $E \to M$.

We introduce one more important C^*-algebra associated with the C^*-algebra A. Namely, consider the tensor product $\mathcal{K} \otimes A$ of the algebra of compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H})$, acting in a Hilbert space $\mathcal{H}$, and the C^*-algebra A. Recall that $\mathcal{K} \otimes A$ is the completion of the algebraic tensor product $\mathcal{K} \odot A$ with respect to a unique C^*-norm having the *cross-norm property*:

$$\|\xi \otimes \eta\| = \|\xi\| \cdot \|\eta\|.$$

Definition 18. The tensor product $A_S := \mathcal{K} \otimes A$ is called the *stabilization* of the C^*-algebra A. A C^*-algebra A is called *stable* if $A_S \cong A$. Two C^*-algebras A and B are *stably equivalent* if $A_S \cong B_S$.

1.3.4 Projectors in C^*-modules

We study now the projectors in C^*-modules. Let $\mathcal{F}$ be a closed submodule in a C^*-module $\mathcal{E}$. It is well known that not any such submodule admits the orthogonal complement. Suppose however that $\mathcal{F}$ has the orthogonal complement denoted by $\mathcal{F}^\perp$ so that $\mathcal{F} \oplus \mathcal{F}^\perp \cong \mathcal{E}$. Then any element $s \in \mathcal{E}$ can be written uniquely in the form

$$s = t + u$$

where $t \in \mathcal{F}$, $u \in \mathcal{F}^\perp$ so that the map $s \mapsto t$ determines a projector $p \in \mathrm{End}_A\mathcal{E}$ with the image, equal to $\mathcal{F}$.

Conversely, if $p \in \mathrm{End}_A\mathcal{E}$ is a projector then the orthogonal complement to $\mathrm{Im}\, p$ exists and coincides with

$$(\mathrm{Im}\, p)^\perp = \mathrm{Im}(1_\mathcal{E} - p) = \mathrm{Ker}\, p.$$

Thus we have established a bijective correspondence between complementable C^*-submodules in $\mathcal{E}$ and images of projectors from $\mathrm{End}_A\mathcal{E}$.

Definition 19. Let A be a unital C^*-algebra. Then the A-compact projectors in $\mathcal{H}_A$ form a subset in the C^*-algebra A_S denoted by $\mathcal{P}(A_S)$. It is a closed subset in the unit ball of A_S.

As examples of projectors in $\mathcal{P}(A_S)$ one can consider the operators of the form

$$P_n = \sum_{j=1}^{n} |e_j\rangle\langle e_j|$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots)$ (with 1 at the jth place).

In conclusion we give a description of A-compact operators in finitely generated projective modules over the algebra A. Consider the modules of the form pA^n where p is a projector in $\mathrm{Mat}_n(A)$.

Proposition 5. *Let p be a projector in $Mat_n(A)$ generating the C^*-module pA^n over the algebra A. Then*

$$\mathcal{K}_A(pA^n) \cong p\, Mat_n(A)p.$$

Proof. Consider the map

$$\mathrm{Fin}_A(pA^n) \longrightarrow p\,\mathrm{Mat}_n(A)p$$

which sends the ketbra-operator of the form $|pa\rangle\langle pb|$ to the matrix with (i,j)-entries equal to

$$\sum_{k,l} p_{ik} a_k b_l^* p_{lj}.$$

This map is an isometric $*$-isomorphism which extends to an isomorphism of $\mathcal{K}_A(pA^n)$ onto $p\,\mathrm{Mat}_n(A)p$. □

1.3.5 Unitary operators. Adjoint operators

We continue to study linear operators in C^*-modules over an algebra A. Recall the definition of the adjoint operator.

Definition 20. Let $T : \mathcal{E} \to \mathcal{F}$ be a bounded A-linear operator acting from a C^*-module $\mathcal{E}$ over the C^*-algebra A to a C^*-module $\mathcal{F}$ over the same algebra. We say that T is *adjointable* or *admits an adjoint operator* if there exists an A-linear operator $T^\dagger : \mathcal{F} \to \mathcal{E}$, called the *adjoint operator* of T, such that

$$(r, Ts) = (T^\dagger r, s)$$

for all $s \in \mathcal{E}$, $r \in \mathcal{F}$.

As in the case of bounded linear operators, acting in a Hilbert space, the adjoint operator is uniquely defined and $T^{\dagger\dagger} = T$. However, in contrast with the bounded operators in a Hilbert case, bounded A-linear operators, acting in C^*-modules, does not always admit the adjoint operators. The reason is that not every closed submodule $\mathcal{F}$ in a C^*-module $\mathcal{E}$ over a C^*-algebra A admits the orthogonal complement $\mathcal{G}$ such that $\mathcal{F} \oplus \mathcal{G} = \mathcal{E}$. (The reader may try to find the corresponding counterexample.)

So we should assume as an extra condition the existence of the adjoint operator. Namely, we shall denote by $\mathrm{Hom}_A(\mathcal{E}, \mathcal{F})$ the vector space of A-linear operators $T : \mathcal{E} \to \mathcal{F}$ which admit the adjoint operator. For $\mathcal{F} = \mathcal{E}$ we denote by $\mathrm{End}_A(\mathcal{E})$ the algebra of A-linear adjointable endomorphisms of $\mathcal{E}$.

Introduce one more class of operators which we shall constantly meet below.

Definition 21. A *unitary operator* $U : \mathcal{E} \to \mathcal{F}$ from a C^*-module $\mathcal{E}$ over a C^*-algebra A to a C^*-module $\mathcal{F}$ over the same algebra is a map $U \in \mathrm{Hom}_A(\mathcal{E}, \mathcal{F})$ such that

$$U^\dagger U = 1_{\mathcal{E}},\ UU^\dagger = 1_{\mathcal{F}}.$$

If such an operator exists then the modules $\mathcal{E}$ and $\mathcal{F}$ are called *unitary equivalent*.

One of the most important properties of C^*-algebras is that the operator algebras over them are again C^*-algebras.

Proposition 6. *Let $\mathcal{E}$ be a (right) C^*-module over a C^*-algebra A. Then the algebra $End_A\mathcal{E}$ of bounded linear adjointable operators in $\mathcal{E}$ is also a C^*-algebra.*

Proof. The norm in the algebra $\mathrm{End}_A\mathcal{E}$ is defined by

$$\|T\| = \sup_{\|s\|\le 1} \|Ts\|$$

where $s \in \mathcal{E}$. By the Cauchy inequality

$$\|Ts\|^2 = \|(s, T^\dagger Ts)\| \le \|s\| \cdot \|T^\dagger Ts\| \le \|T^\dagger T\| \cdot \|s\|^2$$

whence

$$\|T\|^2 \le \|T^\dagger T\| \le \|T^\dagger\| \cdot \|T\|,$$

i.e. $\|T\| \le \|T^\dagger\|$, so $\|T\| = \|T^\dagger\|$ since $T^{\dagger\dagger} = T$. Hence,

$$\|T\|^2 \le \|T^\dagger T\| \le \|T\|^2 \Longrightarrow \|T^\dagger T\| = \|T\|^2,$$

i.e. $\|\cdot\|$ is a C^*-norm.

To proof the completeness of the space $\mathrm{End}_A\mathcal{E}$ we note first of all that a Cauchy sequence of adjointable operators $\{T_n\}$ should converge to some bounded linear operator T. By the above argument the sequence of adjoint operators $\{T_n^\dagger\}$ is also a Cauchy sequence and so converges to some bounded linear operator S. Since

$$(r, Ts) = \lim_n (r, T_n s) = \lim_n (T_n^\dagger r, s) = (Sr, s)$$

for all $r, s \in \mathcal{E}$ the operator T admits an adjoint operator which coincides with S. This proves the completeness of the space $\mathrm{End}_A\mathcal{E}$. □

Corollary 3. *The algebra $\mathcal{K}_A(\mathcal{E})$ which consists of A-compact operators, acting in a C^*-module $\mathcal{E}$, is a C^*-algebra and two-sided ideal in the algebra $End_A\mathcal{E}$.*

In particular, the algebra $\mathcal{K} \equiv \mathcal{K}(\mathcal{H})$ which consists of compact operators, acting in a Hilbert space $\mathcal{H}$, is a two-sided ideal in the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators in $\mathcal{H}$.

1.3.6 Projective C^*-modules

In this section we shall prove an analogue of the Serre–Swan theorem for arbitrary unital C^*-algebras, more precisely, we establish a bijective correspondence between A-compact projectors and finitely generated projective C^*-modules over A.

Theorem 5. *Let A be a unital C^*-algebra. Then the right C^*-modules of the form $p\mathcal{H}_A$ with $p \in \mathcal{P}(A_S)$ are finitely generated projective modules over A. In other words, each of them is isomorphic to the direct summand in a free module A^n for some n.*

Idea of the proof: consists in associating with the module $p\mathcal{H}_A$ an isomorphic module embedded into the free module $P_n\mathcal{H}_A \cong A^n$ where P_n is the standard projector in $\mathcal{H}_A$ introduced in the beginning of Sec.1.3.2. In order to compare the original projector p with P_n we should first "rotate" it with the help of a unitary element $u_n \in \mathrm{End}_A\mathcal{H}_A$ to achieve the inequality $u_n p u_n^* \le P_n$.

Turning to the construction of such operator note that for a given $\varepsilon > 0$ there exists n for which

$$\|p - P_n p P_n\| < \varepsilon/3.$$

The operator $a_n := P_n p P_n$ is positive since

$$a_n \geq a_n^2 = a_n^* a_n \geq 0.$$

Indeed, the inequality $a_n \geq a_n^2$ may be written as the relation $P_n p P_n \geq P_n p P_n p P_n$ which follows from the inequality $p \geq p P_n p$ implied by $I \geq P_n$.

Moreover, $\|a_n - a_n^2\| < \varepsilon$ since

$$\|a_n - a_n^2\| \leq \|a_n - p\| + \|p(p - a_n)\| + \|(p - a_n)a_n\| \leq \\ < \varepsilon/3 + \|p\|\varepsilon/3 + \|a_n\|\varepsilon/3 \leq \varepsilon$$

because $\|p\| \leq 1$ and $\|a_n\| \leq 1$.

It is easy to see that the spectrum of a_n is contained in the interval $[0, 1]$ without the point $1/2$. One can also show that this spectrum is contained in the union of intervals $[0, 2\varepsilon] \cup [1 - 2\varepsilon, 1]$ (assuming that $\varepsilon < 1/4$).

We apply now the spectral theorem for selfadjoint elements of a Banach algebra (cf., e.g. [9]). Denote by p_n the spectral projector associated with the interval $[1 - 2\varepsilon, 1]$. This projector is equal to $f(a_n)$ for any continuous function f on the interval $[0, 1]$ such that $0 \leq f \leq 1$ and

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 2\varepsilon \\ 1 & \text{for } 1 - 2\varepsilon \leq t \leq 1. \end{cases}$$

We have $p_n \in \mathcal{P}(A_S)$ with $\|p_n - a_n\| < 2\varepsilon$ so that $\|p_n - p\| < 3\varepsilon < 3/4$.

We use now the following lemma.

Lemma 5. *If p, q are two projectors in a unital C^*-algebra B satisfying the relation $\|p - q\| < 1$ then there exists a unitary element $u \in B$ such that $q = upu^\dagger$.*

Proof of the lemma. Consider the elements $2p - 1$ and $2q - 1$ from the algebra B. They are unitary selfadjoint elements from B with spectra contained in the set $\{-1, 1\}$. Define

$$2r := (2p - 1)(2q - 1) + 1 \Longrightarrow r = 2qp - p - q + 1.$$

Then $qr = qp = rp$ and

$$r^\dagger rp = r^\dagger qr = (qr)^\dagger r = (rp)^\dagger r = pr^\dagger r,$$

i.e. p commutes with $|r|^2$, hence with $|r|$. Moreover, the element r is invertible in B since

$$\|r - 1\| = \|2qp - q - p\| = \|(q - p)(2p - 1)\| \leq \|q - p\| < 1.$$

Set $u = r|r|^{-1}$. It is a unitary element in B satisfying the condition

$$upu^\dagger = rp|r|^{-2}r^\dagger = q.$$

□

By the above lemma $p_n = u_n p u_n^\dagger$ for some unitary element u_n from the unital C^*-algebra $\mathrm{End}_A \mathcal{H}_A$. Moreover, $p_n \leq P_n$. Indeed, the evident relation

$$P_n a_n = a_n P_n = a_n$$

implies (with the help of approximation of f by polynomials) that

$$P_n p_n = p_n P_n = p_n.$$

It follows that $P_n - p_n$ is a projector in $\mathcal{P}(A_S)$ since

$$(P_n - p_n)^2 = P_n - p_n P_n - P_n p_n + p_n = P_n - p_n.$$

The C^*-module $p_n \mathcal{H}_A$ is finitely generated and projective since it is a direct summand in the free module $P_n \mathcal{H}_A \cong A^n$:

$$p_n \mathcal{H}_A \oplus (P_n - p_n)\mathcal{H}_A = P_n \mathcal{H}_A.$$

The map $s \mapsto u_n s$ establishes a unitary equivalence of A-modules $p\mathcal{H}_A$ and $p_n \mathcal{H}_A$. So the C^*-module $p\mathcal{H}_A$ is also isomorphic to a direct summand in A^n, i.e. it is finitely generated and projective. □

The converse result is also true.

Theorem 6. *Let $\mathcal{E}$ be a (right) finitely generated projective module over a unital C^*-algebra A. Then it can be provided with the structure of a C^*-module over A in such a way that it will be isomorphic to $p\mathcal{H}_A$ for some $p \in \mathcal{P}(A_S)$.*

Proof. The module $\mathcal{E}$ has the form $\mathcal{E} = eA^n$ where e is an idempotent in $\mathrm{Mat}_n(A)$. Since the image of an idempotent is closed the structure of a C^*-module on $\mathcal{E}$ may be defined by restricting the standard structure of C^*-module from A^n to $\mathcal{E}$.

Note that the operator e admits an adjoint operator. Indeed, if $\{u_j\}_{j=1}^n$ is the standard basis in A^n so that

$$1_{A^n} = \sum_{j=1}^n |u_j\rangle\langle u_j|$$

then the idempotent e may be rewritten in its terms in the form

$$e = \sum_{j=1}^n |eu_j\rangle\langle u_j|,$$

whence

$$e^\dagger = \sum_{j=1}^n |u_j\rangle\langle eu_j|.$$

As we have pointed out before,

$$(\mathrm{Im}\, e)^\perp = \mathrm{Ker}\, e^\dagger$$

since $e^\dagger s = 0 \iff (er, s) = (r, e^\dagger s) = 0$ for all $r \in A^n$. It follows that $(\mathrm{Ker}\, e^\dagger)^\perp = \mathrm{Im}\, e$ and, analogously, $(\mathrm{Im}\, e^\dagger)^\perp = \mathrm{Ker}\, e$, $(\mathrm{Ker}\, e)^\perp = \mathrm{Im}\, e^\dagger$ since $e^\dagger$ is also an idempotent.

Now,

$$\operatorname{Ker} e = \operatorname{Ker}(e^\dagger e)$$

since the equality $(e^\dagger e)s = 0$ implies that $(es, es) = (s, (e^\dagger e)s) = 0 \Longrightarrow es = 0$. Switching to the orthogonal complements, we obtain

$$\operatorname{Im} e^\dagger = \operatorname{Im}(e^\dagger e).$$

So if the element $s \in A^n$ then $e^\dagger s = (e^\dagger e)t$ for some $t \in A^n$ and we can represent it as the sum

$$s = et + (s - et) \in \operatorname{Im} e \oplus \operatorname{Ker} e^\dagger = \operatorname{Im} e \oplus (\operatorname{Im} e)^\perp.$$

Thus, we have shown that $A^n = \mathcal{E} \oplus \mathcal{E}^\perp$, i.e. $\mathcal{E}$ admits the orthogonal complement. So there exists a projector $p \in \operatorname{Mat}_n(A)$ such that $\mathcal{E} = pA^n$. Since $A^n \cong P_n(\mathcal{H}_A)$ and $p \leq P_n$ under such identification we have $\mathcal{E} = p\mathcal{H}_A$. □

Remark 6 (Kaplansky formula). There is an explicit formula expressing the projector p through the idempotent e. Namely, consider the operator

$$r = ee^\dagger = (1 - e^\dagger)(1 - e) = 1 + (e - e^\dagger)(e^\dagger - e) = 1 - e^\dagger - e + ee^\dagger + e^\dagger e.$$

It is a positive and invertible element in $\operatorname{Mat}_n(A)$ (since these properties have all elements of the form $1 + a^\dagger a$). Moreover, r commutes both with e and with $e^\dagger$. Indeed,

$$re = er = ee^\dagger e \qquad re^\dagger = e^\dagger r = e^\dagger ee^\dagger.$$

So r^{-1} also commutes with e and $e^\dagger$. We set now $p := ee^\dagger r^{-1}$. Then $p = p^\dagger$ and

$$p^2 = ee^\dagger ee^\dagger r^{-2} = ere^\dagger r^{-2} = p,$$

i.e. p is a projector. We also have, $ep = p$ and $pe = e$ since

$$per = ee^\dagger e = er$$

so that the image of p coincides with the image of e.

The Theorem 6 implies

Corollary 4. *For an arbitrary Hermitian vector bundle $E \to M$ over a compact manifold M the following equality*

$$\mathcal{K}_A(\Gamma(M, E)) = End_A(\Gamma(M, E)) \cong \Gamma(M, End\, E)$$

holds.

1.4 K-theory

1.4.1 K_0-group

We introduce the following equivalence relation on the set of projectors in a C^*-algebra A.

Definition 22. Two projectors $p, q \in \mathcal{P}(A_S)$ are *equivalent* if $q = upu^\dagger = upu^{-1}$ for some unitary element $u \in \mathrm{End}_A \mathcal{H}_A$.

Denote by

$$V^{\mathrm{top}}(A) := \mathcal{P}(A_S)/\sim$$

the quotient of the space $\mathcal{P}(A_S)$ with respect to the introduced equivalence relation.

Proposition 7. *The set $V^{top}(A)$ is a unital commutative semigroup.*

Proof. Define the direct sum of projectors $p, q \in \mathcal{P}(A_S)$ by the formula

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Using this, we define the sum in $V^{\mathrm{top}}(A)$ as

$$[p] + [q] = [p \oplus q].$$

It is correctly defined since

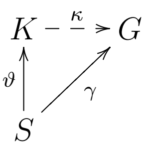
$$upu^{-1} \oplus vqv^{-1} = (u \oplus v)(p \oplus q)(u^{-1} \oplus v^{-1})$$

for unitary $u, v \in \mathrm{End}_A \mathcal{H}_A$. Moreover,

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix},$$

i.e. this semigroup is commutative. The role of the unit (or better say, zero) in this semigroup is played by the zero class [0]. □

Grothendieck construction. We can associate in a canonical way with any unital commutative semigroup S a group K called the *Grothendieck group* of the semigroup S. It is a commutative group provided with a unital semigroup homomorphism $\vartheta : S \to K$ which has the following universal property: if G is any other group provided with a unital semigroup homomorphism $\gamma : S \to G$ then there exists a unique group homomorphism $\kappa : K \to G$ such that the following diagram



commutes, i.e. $\gamma = \kappa \circ \vartheta$.

The group K is uniquely determined up to an isomorphism. It can be constructed in the following way. Consider on the set $S \times S$ the following equivalence relation:

$$(x, y) \sim (x', y') \iff \text{if there exists } z \in S \text{ such that } x + y' + z = x' + y + z.$$

Then the group K is defined as $K = S \times S/\sim$ and the homomorphism ϑ is given by the formula: $\vartheta(x) := [x, 0]$ so that $[x, y] = \vartheta(x) - \vartheta(y)$ in the group K.

Definition 23. The *topological K_0-group* of a unital C^*-algebra A is the Grothendieck group $K_0^{\mathrm{top}}(A)$ of the semigroup $V^{\mathrm{top}}(A)$.

We switch now to the construction of the algebraic K_0-group. For that we prove first the following lemma.

Lemma 6. *Let $e \in Mat_n(\mathcal{A})$ and $f \in Mat_m(\mathcal{A})$ be two matrix idempotents over a unital ring $\mathcal{A}$. Then the associated finitely generated modules $e\mathcal{A}^n$ and $f\mathcal{A}^m$ are isomorphic if and only if there exists an invertible matrix $a \in Mat_N(\mathcal{A})$ with $N > m, n$ such that*

$$a \begin{pmatrix} e & 0 \\ 0 & 0_{N-n} \end{pmatrix} a^{-1} = \begin{pmatrix} f & 0 \\ 0 & 0_{N-m} \end{pmatrix} .$$

Proof. Necessity. Suppose that there exists an isomorphism $\varphi : e\mathcal{A}^n \to f\mathcal{A}^m$. Extending φ by zero to $(1-e)\mathcal{A}^n$, we obtain a morphism $\psi : \mathcal{A}^n \to \mathcal{A}^m$ and, analogously, extending φ^{-1} by zero to $(1-f)\mathcal{A}^m$ we get a morphism $\chi : \mathcal{A}^m \to \mathcal{A}^n$. These morphisms may be written in the form

$$\psi(s) = gs \quad \text{and} \quad \chi(t) = ht$$

for appropriate matrices $g \in \mathrm{Mat}_{m,n}(\mathcal{A})$ and $h \in \mathrm{Mat}_{n,m}(\mathcal{A})$. These matrices satisfy the following easily checked relations:

$$gh = f,\ hg = e \quad \text{and} \quad g = ge = fg,\ h = eh = hf.$$

Set now $N = m + n$ and note that

$$\begin{pmatrix} g & 1-f \\ 1-e & h \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h & 1-e \\ 1-f & g \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$$

and also

$$\begin{pmatrix} g & 1-f \\ 1-e & h \end{pmatrix} \begin{pmatrix} h & 1-e \\ 1-f & g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

This proves the necessity of conditions of the lemma.

Sufficiency. If $a(e \oplus 0)a^{-1} = f \oplus 0$ then $ae\mathcal{A}^N = fa\mathcal{A}^N$. Plugging $\mathcal{A}^N = \mathcal{A}^n \oplus \mathcal{A}^m$ into this relation, we obtain an explicit module isomorphism $\varphi : e\mathcal{A}^n \to f\mathcal{A}^m$. □

Denote by $Q_n(\mathcal{A})$ the set of idempotents in the algebra $\mathrm{Mat}_n(\mathcal{A})$ and by $\mathrm{GL}_n(\mathcal{A})$ the group of invertible elements in $\mathrm{Mat}_n(\mathcal{A})$. There are natural embeddings

$$\mathrm{Mat}_n(\mathcal{A}) \hookrightarrow \mathrm{Mat}_{n+1}(\mathcal{A}), \quad m \longmapsto \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathrm{GL}_n(\mathcal{A}) \hookrightarrow \mathrm{GL}_{n+1}(\mathcal{A}), \quad a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} .$$

The first of them generates also the embedding $Q_n(\mathcal{A}) \hookrightarrow Q_{n+1}(\mathcal{A})$. Using these embeddings we can define the inductive limits

$$\mathrm{Mat}_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathrm{Mat}_n(\mathcal{A}),\ Q_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} Q_n(\mathcal{A}),\ \mathrm{GL}_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathrm{GL}_n(\mathcal{A}).$$

Definition 24. Two idempotents $e, f \in Q_m(\mathcal{A})$ are *equivalent* if they are conjugate in $\mathrm{GL}_\infty(\mathcal{A})$, i.e. if for some n there exists an element $a \in \mathrm{GL}_{n+m}(\mathcal{A})$ such that

$$a \begin{pmatrix} e & 0 \\ 0 & 0_n \end{pmatrix} a^{-1} = \begin{pmatrix} f & 0 \\ 0 & 0_n \end{pmatrix} .$$

Consider the set

$$V^{\mathrm{alg}}(\mathcal{A}) = Q_\infty(\mathcal{A})/\sim$$

which is the quotient of $Q_\infty(\mathcal{A})$ with respect to the introduced equivalence relation. Define the sum in $V^{\mathrm{alg}}(\mathcal{A})$ by the rule

$$[e] + [f] = \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right] = \left[\begin{pmatrix} f & 0 \\ 0 & e \end{pmatrix}\right] .$$

This operation is correctly defined due to the relation $e \oplus f \sim f \oplus e$. Hence, $V^{\mathrm{alg}}(\mathcal{A})$ is a commutative semigroup and we can define the *algebraic K_0-group* $K_0^{\mathrm{alg}}(\mathcal{A})$ as the Grothendieck group of the semigroup $V^{\mathrm{alg}}(\mathcal{A})$.

Theorem 7. *For an arbitrary unital C^*-algebra A both K_0-groups coincide, i.e.*

$$K_0^{top}(A) = K_0^{alg}(A).$$

Proof. Suppose first that idempotents $e, f \in Q_\infty(A)$ are equivalent. By Lemma 6 there exists a sufficiently large n for which the right A-modules eA^n and fA^n are isomorphic. Then by Theorem 6 there exist projectors $p, q \in \mathrm{Mat}_n(A)$ such that

$$eA^n = p\mathcal{H}_A \qquad fA^n = q\mathcal{H}_A.$$

It follows that $p \sim e \sim f \sim q$ in $Q_\infty(A)$, i.e. there exists an element $z \in \mathrm{GL}_\infty(A)$ such that

$$q = zpz^{-1}.$$

In order to prove the equivalence of projectors p and q in the space $\mathcal{P}(A_S)$ we have to find a unitary operator u in $\mathcal{H}_A$ such that $q = upu^{-1}$. For that represent z in the polar form $z = u|z|$ where u is a unitary operator in the space $\mathrm{End}_A\mathcal{H}_A$. Then

$$|z|p|z|^{-1} = u^\dagger qu = (u^\dagger qu)^\dagger = |z|^{-1}p|z|,$$

i.e. p commutes with $|z|^2$, hence also with $|z|$ (why?). So, $q = upu^\dagger$.

Conversely, suppose that projectors $p, q \in \mathcal{P}(A_S)$ are equivalent to each other, i.e. there exists a unitary operator $u \in \mathrm{End}_A\mathcal{H}_A$ such that $q = upu^\dagger$. Then, as in the proof of Theorem 5, we can find for some sufficiently large n projectors $p_n, q_n \in \mathrm{Mat}_n(A)$ such that $p_n \sim p \sim q \sim q_n$ by unitary conjugations which implies that

$$q_n = vp_nv^\dagger \tag{1.9}$$

for some unitary operator $v \in \mathrm{End}_A\mathcal{H}_A$. If $v \in \mathrm{GL}_m(A)$ with some $m \geq n$ then the projectors p_n and q_n belong to the same class in $Q_\infty(A)$, i.e. the modules $p_n\mathcal{H}_A$ and $q_n\mathcal{H}_A$ are isomorphic. But in this case the modules $p\mathcal{H}_A$ and $q\mathcal{H}_A$ are also isomorphic, i.e. by Lemma 6 they belong to the same class in $K_0^{\mathrm{alg}}(A)$.

In the case when v is a general unitary operator in $\mathcal{H}_A$ we can however suppose that $v \in A_S^+$ where A_S^+ denotes the unitalization of the algebra A_S. Indeed, as in the proof of Lemma 5 we can construct unitary operators u_n and v_n, belonging to the group $\mathrm{GL}_n(A) \subset A_S^+$, such that

$$p_n = u_n p u_n^\dagger \quad \text{and} \quad q_n = v_n q v_n^\dagger.$$

The operator v in these terms is written in the form $v = v_n u u_n^\dagger$ (one can check it by plugging this expression for v into (1.9)). This formula implies that $v \in A_S^+$.

Using this fact we can find for a sufficiently large $m \geq n$ a unitary operator $w \in \mathrm{Mat}_m(A)$ approximating v in $\mathrm{Mat}_m(A)$ with the given accuracy $\varepsilon < 1/4$. Denote by $\hat{q}_n \in \mathrm{Mat}_m(A)$ the projector of the form $\hat{q}_n = w p_n w^\dagger$. Then

$$\begin{aligned} q_n - \hat{q}_n = P_m(q_n - \hat{q}_n)P_m = \\ = P_m(v-w)p_n(v-w)^\dagger P_m + P_m w p_n (v-w)^\dagger P_m + P_m(v-w)p_n w^\dagger P_m = \\ = P_m(v-w)P_m p_n P_m (v-w)^\dagger P_m + P_m w P_m p_n P_m (v-w)^\dagger P_m + \\ + P_m(v-w)P_m p_n P_m w^\dagger P_m \end{aligned}$$

where the latter equality follows from the evident relation $p_n = P_m p_n P_m$. It implies that the operator $q_n - \hat{q}_n$ admits the estimate

$$\begin{aligned} \|q_n - \hat{q}_n\| = \|P_m(v-w)P_m p_n P_m (v-w)^\dagger P_m + \\ + P_m w P_m p_n P_m (v-w)^\dagger P_m + P_m(v-w)P_m p_n P_m w^\dagger P_m\| < \\ < \varepsilon^2 + 2\varepsilon < 1. \end{aligned}$$

So by Lemma 5 there exists a unitary operator $w_m \in \mathrm{Mat}_m(A)$ such that $q_n = w_m \hat{q}_n w_m^\dagger$. Then the operator $z_m := w_m w \in \mathrm{GL}_m(A)$ will satisfy the relation $q_n = z_m p_n z_m^{-1}$ which implies that the projectors p_n and q_n belong to the same class in $Q_\infty(A)$, i.e. the modules $p_n\mathcal{H}_A$ and $q_n\mathcal{H}_A$ are isomorphic. But in this case the modules $p\mathcal{H}_A$ and $q\mathcal{H}_A$ are also isomorphic so by Lemma 6 they belong to the same class in $K_0^{\mathrm{alg}}(A)$. □

Due to the proved theorem we shall omit further on the indices "top" and "alg" in the notations of $V(A)$ and $K_0(A)$.

Any element from $K_0(A)$ is represented in the form $[p] - [q]$ where projectors $p, q \in \mathcal{P}(A_S)$. Let $\varphi : A \to B$ be a unitary morphism of C^*-algebras. Denote by $K_0\varphi : K_0(A) \to K_0(B)$ the map given by the formula

$$K_0\varphi : [p] - [q] \longrightarrow [\varphi(p)] - [\varphi(q)].$$

Proposition 8. *The correspondence*

$$(A, \varphi) \longmapsto (K_0(A), K_0\varphi)$$

determines the covariant functor from the category of unital C^*-algebras into the category of Abelian groups.

Examples of K_0-groups

1. $K_0(\mathbb{C}) = \mathbb{Z}$. Indeed, $V(\mathbb{C}) = \mathbb{N}$ since all projectors in $\mathcal{P}(\mathbb{C}_S)$ have finite rank which is their only invariant.

2. Analogously, $K_0(\mathrm{Mat}_n(\mathbb{C})) = \mathbb{Z}$.

3. The group $K_0(\mathcal{L}(\mathcal{H}))$ for the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators in a Hilbert space is equal to zero.

Properties of K_0-functor

1. *stability*: $K_0(A_S) = K_0(A)$.

2. *half-exactness*: K_0 transforms short exact sequences of the form $0 \to J \to A \to B \to 0$ to the sequences

$$K_0(J) \to K_0(A) \to K_0(B)$$

which are exact in the middle term.

3. K_0 *commutes with inductive limits.*

1.4.2 Higher K-groups

In order to define the higher K-groups we introduce the notion of *suspension* of a C^*-algebra A. It is a C^*-algebra of the form

$$\Sigma A := A \otimes C_0(\mathbb{R}) \cong C_0(\mathbb{R}, A)$$

where $C_0(X)$ (resp. $C_0(X, A)$) denotes the space of continuous functions (resp. with values in A) on a locally compact topological space X vanishing at infinity. Using this notion we define the *K-group of order n* for the C^*-algebra A as

$$K_n(A) := K_0(\Sigma^n A).$$

Theorem 8 (Bott periodicity theorem)**.** *For any C^*-algebra A and any natural n we have the following isomorphisms*

$$K_{2n}(A) \cong K_0(A), \quad K_{2n+1}(A) \cong K_1(A).$$

In view of this theorem it is sufficient to study, apart from the group $K_0(A)$, only the group $K_1(A)$ for which we shall give another, equivalent definition.

Namely, we introduce the group

$$K_1^{\mathrm{top}}(A) = [C_0(\mathbb{R}), A_S]$$

identified with the set of homotopy classes of homomorphisms $C_0(\mathbb{R}) \to A_S$. This definition may be rewritten in the form

$$[C_0(\mathbb{R}), A_S] \cong [C(\mathbb{T}), A_S^+]_+$$

where A_S^+ denotes the unitalization of the algebra A_S and the index "+" in the notation $[X,Y]_+$ indicates that we are considering the set of homotopy classes of continuous maps $X \to Y$ of the pointed topological spaces X, Y.

Note that the C^*-algebra $C(\mathbb{T})$ is generated by the unique unitary element $t \mapsto e^{it}$. So a homomorphism from $K_1^{\text{top}}(A) \cong [C(\mathbb{T}), A_S^+]_+$ is determined by the choice of a unitary element in $A_S^+ = (\mathcal{K} \otimes A)^+$. Hence, we can identify $K_1^{\text{top}}(A)$ with group $\pi_0(\mathrm{U}(A_S^+))$ of connected components of the unitary group $\mathrm{U}(A_S^+)$.

Using the fact that $\mathcal{K}(\mathcal{H}) = \varinjlim \mathrm{Mat}_n(\mathbb{C})$, we can rewrite the latter definition of $K_1^{\text{top}}(A)$ in the form

$$K_1^{\text{top}}(A) = \varinjlim \mathrm{U}_n(A)/\mathrm{U}_n(A)^0 = \varinjlim \mathrm{GL}_n(A)/\mathrm{GL}_n(A)^0$$

where $\mathrm{U}_n(A)$ denotes the subgroup in $\mathrm{Mat}_n(A)$ consisting of unitary elements, and $\mathrm{U}_n(A)^0$ is the connected component of identity in $\mathrm{U}_n(A)$.

Examples of topological K_1-groups

1. $K_1^{\text{top}}(\mathbb{C}) = 0$. This fact follows from the connectedness of the group $\mathrm{U}(\mathcal{K}^+)$ (where $\mathcal{K} = \mathcal{K}(\mathcal{H})$) which you may check by yourself.

2. Analogously, $K_1^{\text{top}}(\mathrm{Mat}_n(\mathbb{C})) = 0$.

The multiplication in the group $K_1^{\text{top}}(A)$ is defined by the formula:

$$[u] \cdot [v] = [uv] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right]$$

where the second equality and commutativity of multiplication follow from the chain of homotopies:

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \sim \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}$$

which the reader may check by himself.

We switch to the definition of the algebraic K_1-group. Recall first of all that the *commutant* of an arbitrary group G is its normal subgroup $G' := [G, G]$ generated by the elements of the form $[g, h] := ghg^{-1}h^{-1}$. The quotient

$$G_{\text{ab}} := G/G'$$

is an Abelian group called the *abelianization* of the group G.

Using this notion, we can define the K_1-*group* of an arbitrary ring $\mathcal{A}$ as

$$K_1^{\text{alg}}(\mathcal{A}) = \mathrm{GL}_\infty(\mathcal{A})_{\text{ab}} = \mathrm{GL}_\infty(\mathcal{A})/\mathrm{GL}_\infty(\mathcal{A})'.$$

Examples of algebraic K_1-groups

1. If $\mathcal{A} = \mathbb{Z}$ then $K_1^{\text{alg}}(\mathbb{Z}) = \mathbb{Z}_2$.

2. If $\mathcal{A} = F$ is a field then $K_1^{\text{alg}}(F) = F^\times$ (the group of invertible elements in F).

In particular, if a ring $\mathcal{A}$ is a unital C^*-algebra A then

$$K_1^{\mathrm{alg}}(A) = \mathrm{GL}_\infty(A)_{\mathrm{ab}} = \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)'.$$

On the other hand

$$K_1^{\mathrm{top}}(A) = \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)^0.$$

Since the commutant is contained in the connected component of identity (why?) there is a natural surjective map

$$K_1^{\mathrm{alg}}(A) \longrightarrow K_1^{\mathrm{top}}(A) \tag{1.10}$$

which is, however, in contrast with the case of K_0-groups, not always injective. Indeed, in the case of the C^*-algebra $A = \mathbb{C}$ we have:

$$K_1^{\mathrm{alg}}(\mathbb{C}) = \mathbb{C}^\times \text{ while } K_1^{\mathrm{top}}(\mathbb{C}) = 0.$$

(The map (1.10) is bijective in the case of the so called *stable* C^*-algebras.)

In the sequel we shall denote by $K_1(A)$ the group $K_1^{\mathrm{top}}(A)$.

1.5 Fredholm operators

1.5.1 Topological theory

Consider the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators acting in a Hilbert space $\mathcal{H}$. The ideal $\mathcal{K} \equiv \mathcal{K}(\mathcal{H})$ of compact operators in the algebra $\mathcal{L}(\mathcal{H})$ will play later the role of the set of infinitesimal elements in this algebra. So it is wotrhwhile to study the so called *Kalkin algebra*

$$Q(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

Note that two operators $S, T \in \mathcal{L}(\mathcal{H})$ have the same image in the algebra $Q(\mathcal{H})$ if and only if $S = T + K$ for some compact operator K.

Proposition 9. *A bounded linear operator $F \in \mathcal{L}(\mathcal{H})$ has an invertible image in the algebra $Q(\mathcal{H})$ if and only if there exists an operator $G \in \mathcal{L}(\mathcal{H})$ such that the operators $1 - GF$ and $1 - FG$ are compact. The latter condition is equivalent to the fact that the image $\operatorname{Im} F$ is closed and the kernel $\operatorname{Ker} F$ and cokernel $\operatorname{Coker} F$ of operator F are finite-dimensional.*

Proof. The first equivalence is evident. To prove the second one suppose that there exists an operator $G \in \mathcal{L}(\mathcal{H})$ such that the operators $1-GF$ and $1-FG$ are compact. Assume that we have already proved that $\operatorname{Im} F$ is closed and show that the kernel $\operatorname{Ker} F$ is finite-dimensional. Note that $\operatorname{Ker} F$ is invariant under the operator $1-GF$ since

$$(1 - GF)\xi = \xi - GF\xi = \xi$$

for $\xi \in \operatorname{Ker} F$. The same is true also for the unit ball in the space $\operatorname{Ker} F$ which coincides therefore with the image of the compact operator $1-GF$. The compactness of this ball implies that $\operatorname{Ker} F$ is finite-dimensional.

We prove now that $\operatorname{Im} F$ is closed. For that we choose an operator R of finite rank so that

$$\|(1 - GF) - R\| < 1/2.$$

For $\xi \in \operatorname{Ker} R$ we shall have

$$\|\xi\| - \|\xi - GF\xi\| \leq \|GF\xi\| \leq \|G\| \cdot \|F\xi\|.$$

On the other hand,

$$\|\xi\| - \|\xi - GF\xi\| \geq \|\xi\| - \frac{1}{2}\|\xi\| = \frac{1}{2}\|\xi\|,$$

i.e.

$$\frac{1}{2}\|\xi\| \leq \|G\| \cdot \|F\xi\|.$$

It follows that

$$\|F\xi\| \geq \frac{\|\xi\|}{2\|G\|}$$

hence, the restriction of the operator F to $\operatorname{Ker} R$ has closed image. But the subspace $(\operatorname{Ker} R)^{\perp} = \operatorname{Im} R^{\dagger}$ is finite-dimensional since the operator R has finite rank. So the space

$$\operatorname{Im} F = F(\operatorname{Ker} R) + F((\operatorname{Ker} R)^{\perp})$$

is closed.

To prove that the cokernel $\operatorname{Coker} F$ is finite-dimensional we note, as above, that the space $\operatorname{Ker} F^{\dagger}$ is invariant under the operator $(1-FG)^{\dagger} = 1-G^{\dagger}F^{\dagger}$ which implies its finite-dimensionality. But

$$\operatorname{Coker} F = \mathcal{H}/\operatorname{Im} F \cong \operatorname{Ker} F^{\dagger}$$

and so is also finite-dimensional.

Conversely, if the image $\operatorname{Im} F$ is closed and the subspaces $\operatorname{Ker} F$ and $\operatorname{Coker} F$ are finite-dimensional then we can construct the desired operator G by setting

$$\begin{cases} G(F\xi) = \xi & \xi \in (\operatorname{Ker} F)^{\perp}, \\ G(\xi) = 0 & \xi \in (\operatorname{Im} F)^{\perp} \cong \operatorname{Ker} F^{\dagger}. \end{cases}$$

Indeed, this operator is correctly defined since the map $F : (\operatorname{Ker} F)^{\perp} \to \operatorname{Im} F$ is bijective. Moreover, the operators $1 - GF$ and $1 - FG$ have finite rank, hence they are compact. □

Definition 25. A bounded linear operator $F : \mathcal{H}_1 \to \mathcal{H}_2$ from a Hilbert space $\mathcal{H}_1$ into another Hilbert space $\mathcal{H}_2$ is called *Fredholm* if its image $\operatorname{Im} F$ is closed and the spaces $\operatorname{Ker} F$ and $\operatorname{Coker} F$ are finite-dimensional. The *index of the Fredholm operator* F is equal to

$$\operatorname{ind} F = \dim \operatorname{Ker} F - \dim \operatorname{Coker} F.$$

For $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ the space of Fredholm operators $F : \mathcal{H} \to \mathcal{H}$ is denoted by $\operatorname{Fred} = \operatorname{Fred}(\mathcal{H})$ and provided with the topology of uniform convergence from $\mathcal{L}(\mathcal{H})$. It is a multiplicative semigroup (cf. property 2 below).

Properties of the index:

1. The map ind : Fred $\to \mathbb{Z}$ is continuous.
2. The map ind is a semigroup homomorphism, i.e.
$$\operatorname{ind}(F_1 F_2) = \operatorname{ind} F_1 + \operatorname{ind} F_2.$$
3. The value of the index does not change under compact perturbations, i.e.
$$\operatorname{ind}(F + K) = \operatorname{ind} F$$
for any $K \in \mathcal{K}$.
4. $\operatorname{ind} F = 0 \Longleftrightarrow F$ is a compact perturbation of an invertible operator.
5. The standard right shift operator in the space ℓ^2 is Fredholm and its index is equal to -1.
6. $\operatorname{ind} F = \dim \operatorname{Ker}(F^\dagger F) - \dim \operatorname{Ker}(F F^\dagger)$.
7. $\operatorname{ind} F^\dagger = -\operatorname{ind} F$.

Theorem 9 (Atiyah–Jänich theorem)**.** *For any compact Hausdorff topological space X there exists a group isomorphism*
$$ind : [X, Fred] \longrightarrow K^0(X)$$
where $K^0(X)$ is the Grothendieck group K(Vect(X)) of the semigroup Vect(X) of virtual vector bundles over X. This isomorphism is functorial in the sense that for any continuous map $\varphi : Y \to X$ of compact topological spaces the following relation
$$ind \circ \varphi^* = K^0\varphi \circ ind$$
holds where $\varphi^* : [X, Fred] \to [Y, Fred]$ is the homomorphism generated by the map $F \mapsto F \circ \varphi$ from the space $C(X, Fred)$ into the space $C(Y, Fred)$ and $K^0\varphi : K^0(X) \to K^0(Y)$.

Remark 7. We explain roughly the idea of the proof of this theorem. Let $F : X \to$ Fred, $x \mapsto F_x$, be a continuous map from the space X into the space of Fredholm operators. We should associate with it an element of $K^0(X)$. The first idea is to identify the desired element with the field of virtual vector spaces of the form $x \mapsto [\operatorname{Ker} F_x] - [\operatorname{Coker} F_x]$. But this map in general is not defined since the dimensions of the spaces $\operatorname{Ker} F_x$ and $\operatorname{Coker} F_x$ may change from point to point.

In order to circumvent this difficulty the following method is used. The field of spaces $x \mapsto \operatorname{Ker} F_x$ is replaced by the trivial bundle over X with fibre $\mathcal{H}/V$ where V is a closed subspace in $\mathcal{H}$ of finite codimension such that
$$V \cap \operatorname{Ker} F_x = \{0\}$$
for all $x \in X$. Such V is chosen using the compactness of X as the intersection
$$V = \bigcap_{i=1}^{m} (\operatorname{Ker} F_{x_i})^\perp$$

where $x_1, \dots, x_m \in X$ is an appropriate finite set of points in X. The class $[\mathcal{H}/V]$ of the trivial vector bundle with fibre $\mathcal{H}/V$ in $K^0(X)$ is the required replacement of the field $x \mapsto \operatorname{Ker} F_x$. It is proved next that the union

$$\bigcup_{x\in X} \mathcal{H}/F_x(V)$$

is the total space of some locally trivial vector bundle $W \to X$ over X and the class $[W]$ of this bundle in $K^0(X)$ may be taken as the replacement of the field $x \mapsto \operatorname{Coker} F_x$. Finally it is proved that

$$\operatorname{ind} F = [\mathcal{H}/V] - [W] \in K^0(X).$$

1.5.2 Fredholm operators in C^*-modules

Definition 26. Let $\mathcal{E}, \mathcal{F}$ be right C^*-modules over a C^*-algebra A and $F \in \operatorname{Hom}_A(\mathcal{E}, \mathcal{F})$ is a bounded A-linear operator. The operator F is called A-*Fredholm* if there exists an operator $G \in \operatorname{Hom}_A(\mathcal{F}, \mathcal{E})$ such that $1_{\mathcal{F}} - FG \in \mathcal{K}_A(\mathcal{F})$ and $1_{\mathcal{E}} - GF \in \mathcal{K}_A(\mathcal{E})$. In the case when $\mathcal{E} = \mathcal{F}$ this condition is equivalent to the invertibility of the image of F in the quotient algebra $\operatorname{End}_A(\mathcal{E})/\mathcal{K}_A(\mathcal{E})$. The set of A-Fredholm operators is denoted by $\operatorname{Fred}_A(\mathcal{E}, \mathcal{F})$.

In this definition one can replace the A-compact operators by the operators of A-finite rank according to the following lemma.

Lemma 7. *Let A be a unital C^*-algebra and J is an ideal in A which closure is denoted by $\bar{J}$. Then any element $a \in A$ which is invertible modulo $\bar{J}$ is also invertible modulo J.*

Proof. Denote by b the element of the algebra A such that $1 - ab \in \bar{J}$. Then there exists an element $c \in J$ for which $\|1 - ab - c\| < 1$ and, consequently, $ab + c$ is invertible. Denote by b_1 the element $b_1 := b(ab + c)^{-1}$. Then

$$1 - ab_1 = 1 - ab(ab + c)^{-1} = c(ab + c)^{-1},$$

i.e. belongs to J. An analogous argument shows that there exists an element b_2 such that $1 - b_2 a \in J$. In other words, the element $[a] := a + J$ of the quotient algebra A/J is invertible both from the right and from the left which means that it is invertible in A/J. □

Choosing for $J = \operatorname{Fin}_A(\mathcal{E}, \mathcal{F})$, we obtain that for an A-Fredholm operator $F : \mathcal{E} \to \mathcal{F}$ it always exists an operator $G \in \operatorname{Hom}_A(\mathcal{F}, \mathcal{E})$ such that $1_{\mathcal{F}} - FG \in \operatorname{Fin}_A(\mathcal{F})$ and $1_{\mathcal{E}} - GF \in \operatorname{Fin}_A(\mathcal{E})$.

Remark 8. Note that the image $\operatorname{Im} F$ of an A-Fredholm operator F should not be closed. Take, for example, for A the algebra $A = C(I)$ of functions which are continuous on the unit interval $I = [0, 1]$ and for $\mathcal{E}$ the algebra A itself. Define the operator F by the formula: $Fa(t) := ta(t)$ for $t \in I$. Since the algebra A is unital we have $\mathcal{K}_A(A) \cong A$ so that any operator from $\operatorname{End}_A A$ is A-compact and A-Fredholm. But the image $\operatorname{Im} F$ is not closed since the function $b(t) = \sqrt{t}$ evidently does not belong to $\operatorname{Im} F$ but it belongs to its closure since it can be uniformly approximated by polynomials vanishing at $t = 0$, hence belonging to $\operatorname{Im} F$. (For example, one can approximate it by the Bernstein polynomials $\sum_{k=1}^{n} C_n^k \sqrt{k/n} t^k (n - t)^{n-k}$.)

Regular Fredholm operators

In order to avoid this problem we introduce the notion of a pseudoinverse operator. In the proof of Proposition 9 we have constructed an operator G such that the operators $1 - FG$ and $1 - GF$ are projectors onto $\operatorname{Ker} F$ and $\operatorname{Ker} F^\dagger$ respectively. The operators F and G satisfy the relations: $FGF = F$ and $GFG = G$.

This observation motivates the following definition.

Definition 27. For a given operator $T \in \operatorname{Hom}_A(\mathcal{E}, \mathcal{F})$ its *pseudoinverse* is the operator $S \in \operatorname{Hom}_A(\mathcal{F}, \mathcal{E})$ such that

$$TST = T \quad \text{and} \quad STS = S.$$

In this case the operators TS and ST are idempotents and have closed images. Moreover, they have the following properties:

1. $\operatorname{Ker} ST = \operatorname{Ker} T$;
2. $\operatorname{Im}(1 - ST) = \operatorname{Ker} T$;
3. $\operatorname{Im}(TS) = \operatorname{Im} T$.

Since the notion of pseudoinvertibility is symmetric with respect to S and T we have similar properties obtained by replacing T by S:

4. $\operatorname{Ker} TS = \operatorname{Ker} S$;
5. $\operatorname{Im}(1 - TS) = \operatorname{Ker} S$;
6. $\operatorname{Im}(ST) = \operatorname{Im} S$.

The proof of these properties we leave as an exercise. For instance, for the first of them we have, on one side, that $\operatorname{Ker} T \subset \operatorname{Ker} ST$ and, on the other side, that $STu = 0 \Longrightarrow TSTu = Tu = 0$.

The operators, having pseudoinverses, are called *regular*. It is clear that the usual Fredholm operators are regular.

Suppose that F is a regular A-Fredholm operator with pseudoinverse S. Denote by G the operator existing by Definition 26. Note that

$$(1 - GF)(1 - SF) = 1 - SF$$

since $\operatorname{Im}(1 - SF) = \operatorname{Ker} F$. Hence, the operator $1 - SF \in \mathcal{K}_A(\mathcal{E})$. Analogously, $1 - FS \in \mathcal{K}_A(\mathcal{F})$. So for a regular A-Fredholm operator F we can use as the operator G from the Definition 26 the pseudoinverse of F.

Moreover, the operators $1 - FS$ and $1 - SF$ have closed images. Indeed, denote the idempotent $1 - SF$ by e. We have the following lemma.

Lemma 8. *Let $p \in \mathcal{K}_A(\mathcal{E})$ be an A-compact projector in a C^*-module $\mathcal{E}$ over a unital C^*-algebra A. Then $p \in Fin_A\mathcal{E}$. Moreover, any idempotent $e \in \mathcal{K}_A(\mathcal{E})$ is in fact an operator of A-finite rank.*

Proof. The algebra $p\,\mathcal{K}_A(\mathcal{E})p$ is a unital C^*-algebra in which the role of unit is played by the projector p itself. This algebra contains $p\,\mathrm{Fin}_A\mathcal{E}p$ as a dense ideal. Since any such ideal, being dense, cannot be proper it should contain the unit $1_\mathcal{E}$ which can be represented in the form

$$1_\mathcal{E} = \sum_{k=1}^{n} |r_k\rangle\langle s_k|$$

for some elements $r_k, s_k \in \mathcal{E}$. Then the projector p will be written in the form

$$p = \sum_{k=1}^{n} p|r_k\rangle\langle s_k|p = \sum_{k=1}^{n} |pr_k\rangle\langle ps_k|$$

which implies that p has A-finite rank. More generally, if e is an A-compact idempotent then the Kaplansky formula (cf. Remark 6) will give an A-compact projector p such that $pe = p$. It follows from the first part of the proof that $e \in \mathrm{Fin}_A\mathcal{E}$. □

Return to the operator $1 - SF$ which we have identified with the idempotent e. If p is the projector corresponding to e by the Kaplansky formula then $p \in \mathrm{Fin}_A\mathcal{E}$ as well as e. It implies that the image of the operator $e = 1 - SF$, coinciding with the image of the projector p, is closed. In analogous way one can prove that the image of the operator $1 - FS$ is closed.

1.5.3 Index of A-Fredholm operators

In order to introduce the index of regular Fredholm operators we shall use the following important theorem.

Theorem 10 (Kasparov absorption theorem). *If $\mathcal{E}$ is an arbitrary countably generated C^*-module over an algebra A then*

$$\mathcal{E} \oplus \mathcal{H}_A \cong \mathcal{H}_A$$

as A-modules.

Proof. We shall construct an intertwining operator T between $\mathcal{H}_A$ and $\mathcal{E} \oplus \mathcal{H}_A$ for which the operator T and the adjoint operator $T^\dagger$ have dense images. Suppose for simplicity the the algebra A is unital (the case of a non-unital algebra is treated in [3], Theorem 4.6). Let $\{u_k\}$ be a countable family of unit vectors generating $\mathcal{E}$. Denote by (s_k) the sequence obtained by the "reproduction" of the sequence (u_k) so that every u_k in it is repeated infinite number of times. Moreover, denote by $\{\xi_k\}$ the canonical system of generators of the A-module $\mathcal{H}_A$ (cf. Sec. 1.3.1). Define an A-compact operator T by the formula

$$T = \sum_{n=1}^{\infty} 2^{-n}|s_n\rangle\langle\xi_n| \;\oplus\; 4^{-n}|\xi_n\rangle\langle\xi_n|.$$

Every time when s_n coincides with u_k we have $T(2^n\xi_n) = (u_k \oplus 2^{-n}\xi_n)$. Since such coincidence occurs for an infinite number of values of n it follows that we can find a subsequence of the sequence $\{u_k \oplus 2^{-n}\xi_n\}$ converging to an element $(u_k \oplus 0)$ which belongs to the closure of the image of T. It follows that any element $(0 \oplus \xi_n)$ also belongs to this closure. Since $T^\dagger(0 \oplus \xi_n) = 4^{-n}\xi_n$ the image of the adjoint operator $T^\dagger$ is dense in $\mathcal{H}_A$.

To finish the proof of the theorem we use the following lemma.

Lemma 9. *Let $T \in Hom_A(\mathcal{E}, \mathcal{F})$ be an A-linear operator acting from a C^*-module $\mathcal{E}$ over a C^*-algebra A into a C^*-module $\mathcal{F}$ over the same algebra A. Suppose that the operator T, as well as the adjoint operator $T^\dagger$, have dense images. Then the C^*-modules $\mathcal{E}$ and $\mathcal{F}$ are unitary equivalent.*

Proof. Note, first of all, that in this case the image $\mathrm{Im}(T^\dagger T)$ is dense in $\mathcal{E}$ (why?). It implies that also the image of the operator $|T| := \sqrt{T^\dagger T}$ is dense in $\mathcal{E}$.

Define an A-linear map

$$U : \ \mathrm{Im}\, T \longrightarrow \mathrm{Im}\, |T|$$

by setting $U(Ts) := |T|s$ for $s \in \mathcal{E}$. The constructed operator is isometric since

$$\|Ts\|^2 = (T^\dagger Ts, s) = \||T|s\|^2.$$

Since the subspaces $\mathrm{Im}\, T$ and $\mathrm{Im}\, |T|$ are dense in $\mathcal{E}$ the constructed isometric operator extends in a unique way to a unitary operator in $\mathrm{Hom}_A(\mathcal{E}, \mathcal{F})$. □

The assertion of the theorem follows from the proved lemma. □

Index of A-Fredholm operators

We show now how, using the above theorem, one can define the index of regular A-Fredholm operators. The absorption theorem implies that any C^*-module $\mathcal{E}$ of A-finite rank may be considered as a submodule in $\mathcal{H}_A$ of the form $p\mathcal{H}_A$ with $p \in \mathcal{P}(A_S)$. This projector determines a class $[p]$ in $K_0(A)$ which is associated with the C^*-module $\mathcal{E}$ and denoted by $[\mathcal{E}]$.

Let now F be a regular A-Fredholm operator. Then the C^*-modules $\mathrm{Ker}\, F = \mathrm{Im}(1 - SF)$ and $\mathrm{Ker}\, F^\dagger = \mathrm{Im}(1 - FS)$ have, as we have pointed out above, the A-finite rank and so determine the elements $[\mathrm{Ker}\, F]$ and $[\mathrm{Ker}\, F^\dagger]$ of the group $K_0(A)$. Taking this into account, we can give the following definition.

Definition 28. Let $F \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$ be a regular A-Fredholm operator. Define the *index* of F by the formula

$$\mathrm{ind}\, F := [\mathrm{Ker}\, F] - [\mathrm{Ker}\, F^\dagger] \in K_0(A).$$

The index of a regular A-Fredholm operator has the following properties the proof of which we leave to the reader.

1. If $F \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$ is a regular operator with pseudoinverse S then S is also a regular A-Fredholm operator and $\mathrm{ind}\, S = -\mathrm{ind}\, F$.

2. If $F \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$ is a regular operator with pseudoinverse S then the adjoint operator $F^\dagger$ is also a regular A-Fredholm operator and $\mathrm{ind}\, F^\dagger = -\mathrm{ind}\, F$.

3. If $F_1 \in \mathrm{Fred}_A(\mathcal{E}_1, \mathcal{F}_1)$ and $F_2 \in \mathrm{Fred}_A(\mathcal{E}_2, \mathcal{F}_2)$ are regular operators than their direct sum $F_1 \oplus F_2$ is also a regular operator in $\mathrm{Fred}_A(\mathcal{E}_1 \oplus \mathcal{E}_2, \mathcal{F}_1 \oplus \mathcal{F}_2)$ and $\mathrm{ind}\,(F_1 \oplus F_2) = \mathrm{ind}\, F_1 + \mathrm{ind}\, F_2$.

4. If $F \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$ is a regular operator and operators $U \in \mathrm{End}_A(\mathcal{E})$ and $V \in \mathrm{End}_A(\mathcal{F})$ are invertible then the operators FU and VF are regular A-Fredholm operators and $\mathrm{ind}\, FU = \mathrm{ind}\, VF = \mathrm{ind}\, F$.

5. If $F_1, F_2 \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$ are regular operators and $F_1 - F_2 \in \mathcal{K}_A(\mathcal{E}, \mathcal{F})$ then $\mathrm{ind}\, F_1 = \mathrm{ind}\, F_2$.

Is it possible to define the index of an arbitrary, possibly not regular A-Fredholm operator? The answer to this question is positive and we describe briefly how such index can be defined.

Let us represent the operators $T \in \mathrm{Hom}_A(\mathcal{E}_1 \oplus \mathcal{E}_2, \mathcal{F}_1 \oplus \mathcal{F}_2)$ in the block form so that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where $T_{ij} \in \mathrm{Hom}_A(\mathcal{E}_i, \mathcal{F}_j)$, $i, j = 1, 2$.

To define the index of a general A-Fredholm operator we use the following lemma.

Lemma 10. *Let $\mathcal{E}, \mathcal{F}$ be C^*-modules over a unital C^*-algebra A and $F \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$. Then there exist a natural number $n \in \mathbb{N}$ and a regular operator $\tilde{F} \in \mathrm{Fred}_A(\mathcal{E} \oplus A^n, \mathcal{F} \oplus A^n)$ such that $\tilde{F}_{11} = F$.*

The proof of this lemma may be found in [3], Lemma 4.10.

We introduce now the *index* of an arbitrary *A-Fredholm operator* F setting by definition

$$\mathrm{ind}\, F = \mathrm{ind}\, \tilde{F}.$$

It is necessary to check only the correctness of this definition depending on the choice of a regular extension $\tilde{F}$ which is, in its turn, determined by the operator G. But any other choice of an operator G', for which the operators $1 - G'F$ and $1 - FG'$ are operators of A-finite rank, will lead to an operator $\widetilde{F'}$ differing from $\tilde{F}$ by an operator of A-finite rank. Hence the operator $\widetilde{F'}$ will have the same index as $\tilde{F}$ due to the property 5 above.

The so defined index of general A-Fredholm operators has the properties similar to the index of regular A-Fredholm operators.

1. If $F_1, F_2 \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$ and $F_1 - F_2 \in \mathcal{K}_A(\mathcal{E}, \mathcal{F})$ then $\mathrm{ind}\, F_1 = \mathrm{ind}\, F_2$, i.e. does not change under A-compact perturbations.

2. The set $\mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$ is open in $\mathrm{Hom}_A(\mathcal{E}, \mathcal{F})$ and the index map $\mathrm{ind} : \mathrm{Fred}_A(\mathcal{E}, \mathcal{F}) \to K_0(A)$ is locally constant.

3. If $F \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{F})$ and $G \in \mathrm{Fred}_A(\mathcal{F}, \mathcal{G})$ then $GF \in \mathrm{Fred}_A(\mathcal{E}, \mathcal{G})$ and

$$\mathrm{ind}\,(GF) = \mathrm{ind}\, G + \mathrm{ind}\, F.$$

Theorem 11 (noncommutative Atiyah–Jänich theorem)**.** *If A is a unital C^*-algebra then the map*

$$ind : \pi_0(Fred_A) \longrightarrow K_0(A)$$

where $Fred_A := Fred_A(\mathcal{H}_A)$ is a group isomorphism.

The proof of this theorem may be found in [3], Theorem 4.19.

Remark 9. Theorem 11 is an extension of Atiyah–Jänich Theorem 9, corresponding to the case $A = C(X)$ where X is a compact topological space, since in this case we have a group isomorphism

$$\pi_0(\mathrm{Fred}_{C(X)}) \cong [X, \mathrm{Fred}].$$

1.6 Morita-equivalence

1.6.1 Morita-equivalence of algebras

We shall study the notion of Morita-equivalence first in the case of algebras. Let A and B be two algebras. Denote by $\mathcal{M}_A$, ${}_A\mathcal{M}$ and ${}_A\mathcal{M}_B$ the categories of respectively right A-modules, left A-modules and (AB)-bimodules, i.e. bimodules over the algebras (A, B).

Definition 29. The algebras A and B are called *Morita-equivalent* if the categories $\mathcal{M}_A$ and $\mathcal{M}_B$ are equivalent. It means that there exist an (AB)-bimodule $\mathcal{E}$ and (BA)-bimodule F with the following bimodule isomorphisms

$$\mathcal{E} \otimes_B \mathcal{F} \cong A, \quad \mathcal{F} \otimes_A \mathcal{E} \cong B. \tag{1.11}$$

Here A is considered as an (AA)-bimodule over A with bimodule structure defined by the equality: $a(b)c := abc$ for all $a, b, c \in A$. Analogously, B is considered as a (BB)-bimodule with bimodule structure defined in the same way. The bimodules $\mathcal{E}$ and $\mathcal{F}$ are called the *equivalence bimodules*.

Remark 10. If the algebras A and B are Morita-equivalent then the categories ${}_A\mathcal{M}$ and ${}_B\mathcal{M}$ and also the categories ${}_A\mathcal{M}_A$ and ${}_B\mathcal{M}_B$ are also equivalent.

Example 5. Any unital algebra A is Morita-equivalent to the matrix algebra $B := \mathrm{Mat}_n(A)$. In this case the equivalence bimodules are given by the bimodules consisting of row- and column-vectors. Indeed, denote by $\mathcal{E} = {}^nA$ the space of row-vectors with n entries provided with left $\mathrm{Mat}_n(A)$-action and right A-multiplication. Denote next by $\mathcal{F} = A^n$ the space of columns with n entries provided with the left A-multiplication and right $\mathrm{Mat}_n(A)$-action. Then the isomorphisms from (1.11) will be given by the following formulas:

$$(a_1, \ldots, a_n) \otimes {}^t(b_1, \ldots, b_n) \longmapsto \sum_{k=1}^{n} a_k b_k,$$
$${}^t(b_1, \ldots, b_n) \otimes (a_1, \ldots, a_n) \longmapsto (b_i a_j)_{i,j=1}^{n}.$$

In order to give another definition of equivalence bimodules consider the following algebra homomorphisms generated by an arbitrary (AB)-bimodule $\mathcal{E}$:

$$A \longrightarrow \mathrm{End}_B\mathcal{E}, \quad a \longmapsto L_a,$$
$$B^o \longrightarrow \mathrm{End}_A\mathcal{E}, \quad b \longmapsto R_b$$

where L_a is the operator of left multiplication by a and R_b is the operator of right multiplication by b. In this formula we have denoted by B^o the algebra *opposite* to the algebra B. By definition it is the algebra

$$B^o = \{b^o : \ b \in B\}$$

with multiplication law: $b^o c^o := (cb)^o$ for $b, c \in B$.

Theorem 12 (Morita theorem)**.** *An (AB)-bimodule $\mathcal{E}$ is an equivalence bimodule if and only if it is finitely generated and projective both as a left A-module and right B-module, and the above homomorphisms*

$$A \longrightarrow End_B\mathcal{E}, \quad B^o \longrightarrow End_A\mathcal{E}$$

are in fact isomorphisms.

The proof of this theorem may be found in [1].

Example 6. Let $\mathcal{E}$ be a finitely generated projective left A-module and $B = (\mathrm{End}_A\mathcal{E})^o$. Then the algebras A and B are Morita-equivalent and the equivalence (AB)-bimodule coincides with $\mathcal{E}$. It implies, in particular, that if E is a complex vector bundle over a manifold M then the algebras $A = C(M)$ and $B = \Gamma(\mathrm{End}\, E)$ are Morita-equivalent. This case reduces to the one just considered after setting $\mathcal{E} = \Gamma(E)$.

Before we switch to the Morita-equivalence of C^*-algebras consider as an intermediate step the construction of equivalence bimodules for involutive algebras.

Recall that the *involutive* or $*$-*algebra* is an algebra A provided by an anti-linear map $* : A \to A$ such that

$$(ab)^* = b^* a^*, \quad (a^*)^* = a$$

for all $a, b \in A$.

Let A be a unital $*$-algebra and $\mathcal{E}$ be an arbitrary unital (i.e. multiplication by 1_A is the identity map on $\mathcal{E}$) right A-module. Assume that we have on $\mathcal{E}$ an *A-valued inner product* $(\cdot\,,\cdot)_A$ which is *full*, i.e. for any $a \in A$ there exist elements $s_k, t_k \in \mathcal{E}$, $k = 1, \dots, n$, such that

$$a = \sum_{k=1}^{n} (s_k, t_k)_A.$$

Suppose that A and B are two $*$-algebras and $\mathcal{E}$ is an (AB)-bimodule provided with a full A-valued inner product ${}_A(\cdot\,,\cdot)$ and full B-valued inner product $(\cdot\,,\cdot)_B$. Assume that these inner products are related by the following *associativity condition*: for any $s, t, u \in \mathcal{E}$

$$ {}_A(s,t)u = s(t,u)_B. \tag{1.12}$$

We assert that such bimodule is necessarily finitely generated and projective both as a left A-module and right B-module. Indeed, it follows from the fullness of $\mathcal{E}$ as a right B-module that there exist elements $s_k, t_k \in \mathcal{E}$, $k = 1, \ldots, n$, such that

$$1_B = \sum_{k=1}^{n} (s_k, t_k)_B.$$

Denote by $\{e_k\}_{k=1}^n$ the standard basis of the module of row-vectors A^n and consider the map $P : A^n \to \mathcal{E}$ sending $e_k \mapsto t_k$, $k = 1, \ldots, n$. We assert that this map has a right inverse which will imply that $\mathcal{E}$ is a finitely generated projective left A-module. Indeed, Consider the A-linear map $Q : \mathcal{E} \to A^n$ given by the formula: $Q(s) = \sum_{k=1}^n {}_A(s, s_k) e_k$. Then

$$PQ(s) = \sum_{k=1}^{n} {}_A(s, s_k) t_k = \sum_{k=1}^{n} s(s_k, t_k)_B = s.$$

An analogous proof shows that $\mathcal{E}$ is a finitely generated projective right B-module.

In fact one can assert that $\mathcal{E}$ is an equivalence bimodule. For the proof denote by $\overline{\mathcal{E}}$ the complex vector space which is complex dual to $\mathcal{E}$ with elements denoted by $\bar{s}$ where $s \in \mathcal{E}$. Then $\overline{\mathcal{E}}$ is a (BA)-bimodule with multiplication defined by the equality: $b\bar{s}a := \overline{a^* s b^*}$ for $s \in \mathcal{E}$, $a \in A$, $b \in B$. This bimodule may be provided also with A-valued and B-valued inner products by the formula

$${}_B(\bar{s}, \bar{t}) := (s, t)_B, \quad (\bar{s}, \bar{t})_A := {}_A(s, t)$$

for $s, t \in \mathcal{E}$.

Consider the bimodule maps

$$f : \mathcal{E} \otimes \mathcal{E}^* \longrightarrow A, \quad s \otimes \bar{t} \longmapsto {}_A(s, t),$$
$$g : \mathcal{E}^* \otimes \mathcal{E} \longrightarrow B, \quad \bar{s} \otimes t \longmapsto (s, t)_B.$$

Both maps are surjective due to the fullness of inner products. It may be shown also that they are isomorphisms (cf. [1], proposition 4.4). Hence, $\mathcal{E}$ is an equivalence bimodule and the algebras A and B are Morita-equivalent.

1.6.2 Morita-equivalence of C^*-algebras

Suppose that A and B are two C^*-algebras and $\mathcal{E}$ is a right C^*-module over the algebra A and $\mathcal{F}$ is the right C^*-module over the algebra B. We shall assume that it is given a representation of the C^*-algebra A in the module $\mathcal{F}$, i.e. a homomorphism $\rho : A \to \mathrm{End}_B \mathcal{F}$. In Sec. 1.3.5 it was shown that in this case one can define the C^*-module over B which is the tensor product $\mathcal{E} \otimes_\rho \mathcal{F} = \mathcal{E} \otimes_A \mathcal{F}$ of C^*-modules $\mathcal{E}$ and $\mathcal{F}$.

Definition 30. A C^*-module $\mathcal{E}$ over a C^*-algebra A is called *full* if the ideal

$$I =: (\mathcal{E}, \mathcal{E}) = \mathrm{span}\{(s, t) : \ s, t, \in \mathcal{E}\}$$

is dense in A.

With this definition we can introduce the notion of the equivalence bimodule for C^*-algebras.

Definition 31. Let A and B are two C^*-algebras. Then the (AB)-*equivalence bimodule* is an (AB)-bimodule $\mathcal{E}$ with the following properties:

1. $\mathcal{E}$ is a left full C^*-module over A and simultaneously a right full C^*-module over B;

2. for all $s, t, u \in \mathcal{E}$ the following *associativity condition* is satisfied:

$$_A(s,t)u = s(t,u)_B$$

where $_A(\cdot\,,\cdot)$ and $(\cdot\,,\cdot)_B$ are the pairings in $\mathcal{E}$ considered as a left C^*-module over A and right C^*-module over B respectively.

Having the notion of equivalence bimodule we can define the Morita equivalence of C^*-algebras.

Definition 32. We call two C^*-algebras A and B *Morita-equivalent* if for them there exists an equivalence (AB)-bimodule $\mathcal{E}$.

It follows from the conditions of associativity and fullness that in this case the operators L_a of left multiplication by elements $a \in A$ and R_b of right multiplication by elements $b \in B$, defined on the C^*-module $\mathcal{E}$, admit adjoint operators. Indeed, let us check it for the operator L_a. For all $s, t, u \in \mathcal{E}$, $a \in A$, $b \in B$ we have:

$$u(as,t)_B = {}_A(u,as)t = {}_A(u,s)a^*t = u(s,a^*t)_B$$

which implies, due to the fullness of $\mathcal{E}$, that $(as,t)_B = (s,a^*t)_B$. Hence, the operator L_a has the adjoint operator $L_a^\dagger = L_{a^*}$. In the same way one can prove that $R_b^\dagger = R_{b^*}$.

Thus, we have correctly defined the representations of C^*-algebras given by operators L_a and R_b:

$$L : A \longrightarrow \mathrm{End}_B\mathcal{E}, \quad R : B^o \longrightarrow \mathrm{End}_A\mathcal{E}.$$

We take this property as the definition of the following notion generalizing that of equivalence bimodule.

Definition 33. Let A and B be two C^*-algebras. Then an (AB)-*correspondence* between the algebras A and B is a homomorphism $\varphi : A \to \mathrm{End}_B\mathcal{E}$ for some right C^*-module $\mathcal{E}$ over B.

Every equivalence (AB)-bimodule, according to the above argument, determines also an (AB)-correspondence between the C^*-algebras A and B.

If $\mathcal{E}$ is a right C^*-module over the algebra A, $\mathcal{F}$ is a right C^*-module over the algebra B and $\varphi : A \to \mathrm{End}_B\mathcal{E}$ is an arbitrary $*$-homomorphism then $\mathcal{F}$ inherits the structure of (AB)-bimodule and we can define the tensor product $\mathcal{E} \otimes_\varphi \mathcal{F} = \mathcal{E} \otimes_A \mathcal{F}$ which is a C^*-module over B. This tensor product operation is associative. In fact, one can introduce an additive category with C^*-algebras as objects and morphisms given by the correspondences between them determined up to isomorphisms.

Using the notion of correspondence it is possible to give an equivalent definition of Morita-equivalence which is close to that used in the algebraic case.

Note, first of all, that any C^*-algebra A is Morita-equivalent to itself since in this case one can take for the equivalence (AA)-bimodule the algebra A considered as a bimodule provided with the pairings

$$_A(a,b) := ab^*, \quad (a,b)_A := a^*b$$

for $a, b \in A$.

Using this, one can show that two C^*-algebras A and B are Morita-equivalent if and only if there exist an (AB)-correspondence $\mathcal{E}$ and (BA)-correspondence $\mathcal{F}$ such that

$$\mathcal{E} \otimes_B \mathcal{F} \cong A, \quad \mathcal{F} \otimes_A \mathcal{E} \cong B.$$

Remark 11. If $\mathcal{E}$ is an equivalence (AB)-bimodule then one can take for $\mathcal{F}$ the *adjoint bimodule* $\overline{\mathcal{E}}$ consisting of elements $\bar{s}$ with $s \in \mathcal{E}$. Then, as it was pointed out before, $\overline{\mathcal{E}}$ is a (BA)-bimodule with multiplication given by the equality: $b\bar{s}a := \overline{a^*sb^*}$ for any $\bar{s} \in \overline{\mathcal{E}}$, $a \in A$, $b \in B$, and pairings given by

$$_B(\bar{s},\bar{t}) := (t,s)_B, \quad (\bar{s}\bar{t})_A := {}_A(s,t).$$

Then $\overline{\mathcal{E}}$ becomes an equivalence (BA)-bimodule and

$$\mathcal{E} \otimes_B \overline{\mathcal{E}} \cong A, \quad \overline{\mathcal{E}} \otimes_A \mathcal{E} \cong B.$$

Remark 12. Morita-equivalence is the equivalence relation. The proof of this assertion we leave as an exercise.

Remark 13. If two C^*-algebras A and B are Morita-equivalent and $\mathcal{E}$ and $\mathcal{F}$ are their equivalence bimodules then we can identify the category $\mathcal{M}_A$ of right C^*-modules over A with the category $\mathcal{M}_B$ of right C^*-modules over B. Indeed, the map $\mathcal{S} \mapsto \mathcal{S} \otimes_A \mathcal{F}$ associates with a right A-module $\mathcal{S}$ the right B-module $\mathcal{S} \otimes_A \mathcal{F}$ and, conversely, the map $\mathcal{T} \mapsto \mathcal{T} \otimes_B \mathcal{E}$ associates with a right B-module $\mathcal{T}$ the right A-module $\mathcal{T} \otimes_B \mathcal{E}$.

We formulate without proof several criterions of Morita-equivalence (their proofs may be found in the book [3], Sec.4.5).

Proposition 10. *Two C^*-algebras A and B are Morita-equivalent if and only if there exist a full right C^*-module $\mathcal{E}$ over A such that $\mathcal{K}_A(\mathcal{E}) \cong B$.*

Proposition 11. *Any C^*-algebra A is Morita-equivalent to its stabilization, i.e. $A_S \cong \mathcal{K} \otimes A$ is Morita-equivalent to A.*

Corollary 5. *If A and B are two stable equivalent C^*-algebras then the algebra A is Morita-equivalent to the algebra B.*

Theorem 13 (Exel)**.** *If C^*-algebras A and B are Morita-equivalent then $K_0(A) \cong K_0(B)$.*

Remark 14. If C^*-algebras A and B are Morita-equivalent then using the GNS-construction one can establish a bijective correspondence between the irreducible representations of algebras A and B.

Chapter 2

ANALYSIS

2.1 Noncommutative integral

2.1.1 Ideals in the algebra of compact operators

For the construction of the noncommutative version of the analysis on C^*-algebras we have to introduce, first of all, the notion of "infinitesimal" elements. In the algebra of bounded linear operators in a Hilbert space their role is played by compact operators for which the degree of "smallness" is measured by the rate of decreasing of their singular values. Let us consider these notions in more detail.

Let T be a compact operator in a Hilbert space $\mathcal{H}$ and $|T| = \sqrt{T^\dagger T}$. Denote by $\{\mu_n(T)\}$ the sequence of the *singular values* (s-values) of operator T given by the eigenvalues of operator $|T|$ in the decreasing order:

$$\mu_0(T) \geq \mu_1(T) \geq \dots$$

so that $\mu_n(T) \to 0$ for $n \to \infty$.

The singular values of operator T may be found by the *minimax principle*, more precisely, they are given by the formula

$$\mu_n(T) = \inf_E \{\| T | E^\perp \| : \ \dim E = n\}$$

so that $\mu_n(T)$ coincides with the infimum of the norms of restrictions of T to orthogonal complements of various n-dimensional subspaces $E \subset \mathcal{H}$. In fact, this infimum is attained on the subspace E_n generated by the first n eigenvectors of operator $|T|$ corresponding to the eigenvalues $\mu_0, \dots, \mu_{n-1}$.

In a different way, one can define $\mu_n(T)$ as the distance from the operator T to the subspace Fin_n of operators of rank$\leq n$. Namely,

$$\mu_n(T) = \inf_R \{\|T - R\| : \ R \in \mathrm{Fin}_n\}.$$

The singular values of operator T have the following properties (cf. [10]).

Properties of s-values:

1. $|\mu_n(T_1) - \mu_n(T_2)| \leq \|T_1 - T_2\|$, in particular, the functional $\mu_n(T)$ depends continuously on T in the uniform topology for any n.

2. $\mu_{n+m}(T_1 + T_2) \leq \mu_n(T_1) + \mu_m(T_2)$ where we use the embedding $\mathrm{Fin}_n + \mathrm{Fin}_m \subset \mathrm{Fin}_{n+m}$.

3. $\mu_{n+m}(T_1 T_2) \leq \mu_n(T_1) \cdot \mu_m(T_2)$.

4. Since $\mu_0(T) = \|T\|$ we have

$$\mu_n(T_1 T_2) \leq \mu_n(T_1)\|T_2\|, \quad \mu_n(T_1 T_2) \leq \|T_1\|\mu_n(T_2).$$

Definition 34. Let T be a compact operator in a Hilbert space $\mathcal{H}$. We say that T belongs to the space $\mathcal{L}^p = \mathcal{L}^p(\mathcal{H})$, $1 \leq p < \infty$, if

$$\sum_{n=0}^{\infty} \mu_n(T)^p < \infty.$$

The space $\mathcal{L}^p$ is an ideal in the algebra $\mathcal{K}$ of compact operators and in the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators in $\mathcal{H}$. We are especially interested in the class $\mathcal{L}^1$ of *nuclear operators* provided with the norm

$$\|T\|_1 := \mathrm{Tr}|T| = \sum_{n=0}^{\infty} \mu_n(T) = \sum_{n=0}^{\infty} (u_n, T u_n)$$

where $\{u_n\}_{n=0}^{\infty}$ is an orthonormal basis in $\mathcal{H}$. (Note that this definition does not depend on the choice of an orthonormal basis in $\mathcal{H}$.)

We introduce now a quantity playing an important role in the sequel:

$$\sigma_N(T) := \sum_{n=0}^{N-1} \mu_n(T).$$

In a different way it may be defined as

$$\sigma_N(T) = \sup_E \{\|T P_E\|_1 : \ \dim E = N\}$$

where P_E is the orthogonal projector to the subspace E, and the supremum is attained again on the subspace E_N generated by the first N eigenvectors of operator T.

It follows from the latter definition that σ_N satisfies the triangle inequality

$$\sigma_N(T_1 + T_2) \leq \sigma_N(T_1) + \sigma_N(T_2)$$

hence, it determines a norm on $\mathcal{K}$.

We shall give one more definition of this quantity which is used below.

$$\sigma_N(T) = \inf\{\|R\|_1 + N\|S\| : \ R, S \in \mathcal{K}, R + S = T\}.$$

It allows to extend the definition of the function σ_N as a function of natural parameter N to arbitrary nonnegative values $\lambda \in [0, \infty)$ by setting

$$\sigma_\lambda(T) = \inf\{\|R\|_1 + \lambda\|S\| : \ R, S \in \mathcal{K}, R + S = T\}.$$

It can be shown that the function $\sigma_\lambda(T)$ has the following properties:

1. the function $\sigma_\lambda(T)$ is piecewise linear and convex; moreover, if $\lambda = N + t$ with $0 \le t < 1$, so that $[\lambda] = N$, then
$$\sigma_\lambda(T) = (1 - t)\sigma_N(T) + t\sigma_{N+1}(T);$$
2. $\sigma_\lambda(S + T) \le \sigma_\lambda(S) + \sigma_\lambda(T)$;
3. if operators S, T are positive then
$$\sigma_{\lambda+\mu}(S + T) \ge \sigma_\lambda(S) + \sigma_\mu(T).$$

The last two properties imply that the inequality

$$\sigma_\lambda(S + T) \le \sigma_\lambda(S) + \sigma_\lambda(T) \le \sigma_{2\lambda}(S + T) \tag{2.1}$$

holds for arbitrary compact positive operators S, T. This subadditivity property of the functional $\sigma_\lambda(T)$ on the cone of positive compact operators will play an important role in the definition of Dixmier trace in the next section.

Apart from ideals $\mathcal{L}^p$ we introduce also the interpolation ideals $\mathcal{L}^{p,q}$.

Definition 35. An operator $T \in \mathcal{L}^{p,q}$ if

$$\sum_{N=1}^{\infty} N^{(\alpha-1)q-1}\sigma_N(T)^q < \infty$$

where $\alpha = 1/p$. Extend this definition to $q = \infty$ by stating that $T \in \mathcal{L}^{p,\infty}$ if the sequence of numbers $\{N^{\alpha-1}\sigma_N(T)\}_{N=1}^{\infty}$ is bounded.

Proposition 12. *Each of the introduced spaces $\mathcal{L}^{p,q}$ is a two-sided ideal in the algebra $\mathcal{K}$ of compact operators. For $p_1 < p_2$ and for $p_1 = p_2$, $q_1 < q_2$ there are the following inclusions*

$$\mathcal{L}^{p_1,q_1} \subset \mathcal{L}^{p_2,q_2}.$$

The proof is left as an exercise.

Let us consider in more detail some particular examples of the spaces $\mathcal{L}^{p,q}$.

The space $\mathcal{L}^{p,p}$, $1 \le p < \infty$, coincides with the space $\mathcal{L}^p$, introduced above, with the norm given by the formula

$$\|T\|_p = (\mathrm{Tr}|T|^p)^{1/p} = \left[\sum_{n=0}^{\infty} \mu_n(T)^p\right]^{1/p}.$$

The space $\mathcal{L}^{p,\infty}$, $1 < p < \infty$, consists of compact operators T for which $\sigma_N(T) = O(N^{1-\alpha})$, i.e. $\mu_n(T) = O(n^{-\alpha})$. There is a natural norm on this space given by

$$\|T\|_{p,\infty} = \sup_N \frac{1}{N^{1-\alpha}} \sigma_N(T).$$

The space $\mathcal{L}^{p,1}$ consists of compact operators T for which the series

$$\sum_{N=1}^{\infty} N^{\alpha-2} \sigma_N(T)$$

is converging which is equivalent to the convergence of the series $\sum_{n=1}^{\infty} n^{\alpha-1} \mu_{n-1}(T)$.

Between these spaces there are the following embeddings:

$$\mathcal{L}^{p-} \equiv \mathcal{L}^{p,1} \subset \mathcal{L}^{p} \equiv \mathcal{L}^{p,p} \subset \mathcal{L}^{p,\infty} \equiv \mathcal{L}^{p+}.$$

We extend the definition of the spaces $\mathcal{L}^{p,q}$, $1 < p < \infty$, $1 \leq q < \infty$, to $p = 1$, $q = \infty$ by setting

$$\mathcal{L}^{1,\infty} = \{T \in \mathcal{K} : \ \sigma_N(T) = O(\log N)\}$$

and providing this space with the norm

$$\|T\|_{1,\infty} = \sup_{N \geq 2} \frac{\sigma_N(T)}{\log N}.$$

If this norm is finite it implies the estimate of the form $\mu_n = O(1/n)$ on the s-values of operator T. The space $\mathcal{L}^{1,\infty}$ is an ideal dual to the ideal

$$\mathcal{L}^{\infty,1} = \{T \in \mathcal{K} : \ \sum_{n=1}^{\infty} \frac{\mu_n(T)}{n} < \infty\}.$$

As we have pointed out before, for the nuclear operators $T \in \mathcal{L}^1$ we can define the trace given by the sum of s-values of this operator. In the sequel the trace will play the role of the noncommutative integral but the trace defined on the class of nuclear operators does not suit this goal unlike the Dixmier trace, introduced in the next section, which is defined on a larger class of operators $T \in \mathcal{L}^{1,\infty}$.

2.1.2 Dixmier trace

Let T be a positive operator belonging to the ideal $\mathcal{L}^{1,\infty}$. We would like to define its trace by the formula

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T) = \lim_{N\to\infty} \frac{\sigma_N(T)}{\log N}. \tag{2.2}$$

Then we are met with two questions:

1. If the limit in Formula (2.2) does exist?
2. If the functional given by the Formula (2.2) is linear?

Note that the problem of linearity of this functional is closely related to the existence of the limit in Formula (2.2). Indeed, to prove the linearity we should compare the quantity

$$\gamma_N = \frac{\sigma_N(T_1 + T_2)}{\log N}$$

with the sum of quantities

$$\alpha_N = \frac{\sigma_N(T_1)}{\log N} \quad \text{and} \quad \beta_N = \frac{\sigma_N(T_2)}{\log N}.$$

The triangle inequality for $\sigma_N(T)$ implies that $\gamma_N \leq \alpha_N + \beta_N$ and from the inequality $\sigma_N(T_1) + \sigma_N(T_2) \leq \sigma_{2N}(T_1 + T_2)$, mentioned in Sec. 2.1.1, we deduce that

$$\alpha_N + \beta_N \leq \frac{\log(2N)}{\log N} \gamma_N.$$

Since $\log(2N)/\log N \to 1$ for $N \to \infty$ we see that the existence of the limit in Formula (2.2) will imply the linearity of the functional (2.2).

Turning to the question of existence of the limit in Formula (2.2) note that for any $T \in \mathcal{L}^{1,\infty}$ the sequence of numbers

$$\left\{ \frac{\sigma_N(T)}{\log N} \right\}$$

is bounded.

It allows to treat the problem of existence of the limit in Formula (2.2) in the following, more general setting. Namely, we look for a linear form on the space $\ell^\infty(\mathbb{N})$ of bounded sequences $a = \{a_n\}_{n=1}^\infty$ denoted by

$$\ell \equiv \mathrm{Lim}_\omega$$

which satisfies the following conditions:

1. $\mathrm{Lim}_\omega a \geq 0$ if all $a_n \geq 0$;
2. $\mathrm{Lim}_\omega a = \lim_{n\to\infty} a_n$ if the limit in the right hand side exists;
3. $\mathrm{Lim}_\omega(a_1, a_1, a_2, a_2, a_3, a_3, \ldots) = \mathrm{Lim}_\omega\{a_n\}$.

The only non-evident condition is the last one which is treated as the *asymptotic scale invariance*. In order to explain the origin of this term let us switch from the sequences $\{a_n\}$ to the functions of a real parameter as we have already done in the case of the function σ_N. Namely, let us associate with the sequence $\{a_n\}_{n=1}^\infty$ a bounded function $f_a(\lambda)$ on the real line defined in the following way: if $\lambda = N + t$ with $0 \leq t < 1$, i.e. $[\lambda] = N$, then we set $f_a(\lambda) = (1-t)a_N + ta_{N+1}$. Thus, the introduced function $f_a(\lambda)$ is piecewise linear.

Replace now the function $f(\lambda) \equiv f_a(\lambda)$ with its *Cesaro average*

$$(Mf)(\lambda) = \frac{1}{\log \lambda} \int_3^\lambda \frac{f(t)}{t}\, dt.$$

This average on bounded functions f has the following property of asymptotic scale invariance:

$$|M(S_\mu f)(\lambda) - MF(\lambda| \longrightarrow 0 \quad \text{for } \lambda \to +\infty$$

where $(S_\mu f)(\lambda) := f(\lambda\mu)$ for any $\mu > 0$. Returning to the property (3), we can interpret it as the asymptotic scale invariance of the function $f_{\tilde{a}} = S_{1/2}(f_a)$ associated with the sequence $\tilde{a} = (a_1, a_1, a_2, a_2, a_3, a_3, \ldots)$.

We shall formulate now more precisely what kind of the limit we want to have on the considered space $C_b(R_+)$ of bounded continuous functions on the halfline $R_+ := [1, \infty)$. Since we are interested only in the limits of such functions at infinity we replace the space $C_b(R_+)$ by the quotient $B_\infty := C_b(R_+)/C_0(R_+)$ modulo the subspace $C_0(R_+)$ of functions vanishing at infinity.

Fix a positive linear form ω on the space $C_b(R_+)$ such that $\omega = 0$ on the subspace $C_0(R_+)$ and $\omega(1) = 1$. In other words, ω is a state on the C^*-algebra B_∞. We can consider $\omega(f)$ as a "generalized limit" of function $f \in C_b(R_+)$ at infinity. Using the form ω, we can define the limit $\mathrm{Lim}_\omega(a)$ of a sequence $a \in \ell^\infty(\mathbb{N})$ by the formula:

$$\mathrm{Lim}_\omega(a) := \omega(Mf_a).$$

Hence, we come to the following definition.

Definition 36. For any state ω on the C^*-algebra $B_\infty = C_b(R_+)/C_0(R_+)$ the *Dixmier trace* of a positive operator $T \in \mathcal{L}^{1,\infty}$ is defined by the formula

$$\mathrm{Tr}_\omega(T) = \mathrm{Lim}_\omega \frac{\sigma_\lambda(T)}{\log \lambda}\,.$$

Properties of Dixmier trace:

1. *Additivity:* $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$.

2. *Positivity:* the Dixmier trace may be extended to the whole ideal $\mathcal{L}^{1,\infty}$ so that the following property $\mathrm{Tr}_\omega(T) \geq 0$ will hold on positive operators $T \in \mathcal{L}^{1,\infty}$.

3. *Unitary equivalence:* $\mathrm{Tr}_\omega(UTU^*) = \mathrm{Tr}_\omega(T)$ for any unitary operator U.

4. *Commutativity:* for any bounded operator $S \in \mathcal{L}(\mathcal{H})$ and any $T \in \mathcal{L}^{1,\infty}$ the following inequality $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$ holds.

5. The trace $\mathrm{Tr}_\omega(T)$ vanishes on the subspace $\mathcal{L}_0^{1,\infty}$ coinciding with the closure of the space Fin of finite rank operators with respect to the norm $\|\cdot\|_{1,\infty}$. In particular, this trace vanishes on all nuclear operators from the space $\mathcal{L}^1$.

We have already explained why the introduced trace should be additive (a detailed proof of this fact cf. in [3], Lemma 7.14). We leave the proof of other properties to the reader as an exercise.

In general, the trace Tr_ω depends on the choice of the state ω, however the following proposition is true.

Proposition 13. *The subspace*

$$\mathcal{N} = \{T \in \mathcal{L}^{1,\infty} : \ Tr_\omega(T) \text{ does not depend on } \omega\}$$

is a closed linear subspace in $\mathcal{L}^{1,\infty}$. *This subspace contains the subspace* $\mathcal{L}_0^{1,\infty}$ *and is closed under conjugation by invertible operators from* $\mathcal{L}(\mathcal{H})$.

Definition 37. We call an operator $T \in \mathcal{L}^{1,\infty}$ *measurable* if there exists the limit

$$\lim_{\lambda\to\infty} \frac{\sigma_\lambda(T)}{\log\lambda} = \lim_{n\to\infty} \frac{\sigma_n(T)}{\log n} .$$

In this case the Dixmier trace $\mathrm{Tr}_\omega(T)$, of course, does not depend on ω and we denote it by

$$\mathrm{Tr}^+ T = \lim_{\lambda\to\infty} \frac{\sigma_\lambda(T)}{\log\lambda}$$

in order to distinguish from the usual trace of nuclear operators.

Example 7. Consider as an example the formula for the trace of Laplace operator on the sphere S^n provided with the standard metric. The eigenvalues of this operator are equal to $l(l+n-1)$, where l is a nonnegative integer, with multiplicities $m_l = \binom{l+n}{n} - \binom{l+n-2}{n}$. The operator $\Delta^{-n/2}$ is measurable and its Dixmier trace is equal to

$$\mathrm{Tr}^+ \Delta^{-n/2} = \frac{2}{n!} .$$

2.1.3 Pseudodifferential operators

Before we compute the Dixmier trace of concrete pseudodifferential operators recall briefly their basic properties (more about pseudodifferential operators cf. in the books [4],[11]).

The pseudodifferential operators generalize the notion of the usual differential operators. Recall that a *differential operator* of degree d in a domain $U \subset \mathbb{R}^n$ is given by the formula

$$P(x,D) = \sum_{|\alpha|\le d} a_\alpha(x) D^\alpha$$

where the coefficients $a_\alpha(x)$ are smooth functions in U. Here, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the multiindex $\alpha \in \mathbb{Z}_+$ with nonnegative integer components and $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ where $D_j = -i\partial/\partial x_j$. Using the Fourier transform we can rewrite this operator in the form

$$P(x,D) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} p(x,\xi) f(y)\, dy\, d\xi \tag{2.3}$$

where

$$p(x,\xi) = \sum_{|\alpha|\le d} a_\alpha(x) \xi^\alpha$$

is the *symbol* of operator $P(x,D)$.

In order to extend this definition to pseudodifferential operators we enlarge the class of admissible symbols. Namely, introduce the class of *symbols* $S^d(U)$ of degree

d which consists of functions $p(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ satisfying on every compact subset $K \subset U$ the estimate: for any multiindices $\alpha, \beta \in \mathbb{Z}_+$ we have

$$\left|D_x^\beta D_\xi^\alpha p(x,\xi)\right| \le C\left(1+|\xi|^2\right)^{\frac{d-|\alpha|}{2}}$$

for all $x \in K$, $\xi \in \mathbb{R}^n$ with the constant C, depending on α, β and K.

Definition 38. A *pseudodifferential operator* of degree d in a domain $U \subset \mathbb{R}^n$ is the operator P given by Formula (2.3) with symbol $p \in S^d(U)$. The space of all such operators is denoted by $\Psi^d(U)$.

This definition implies that the operator P is correctly defined as a linear operator acting continuously from the space $\mathcal{D}(U)$ of C^∞-smooth functions with compact supports in U to the space $\mathcal{E}(U) \equiv C^\infty(U)$ of C^∞-smooth functions in the domain U. By duality it can be extended to a linear continuous operator $P : \mathcal{E}'(U) \to \mathcal{D}'(U)$ acting on the distributions with compact support in the domain U. If, in particular, $U = \mathbb{R}^n$, then such operator extends to a linear continuous operator acting in the Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions.

The *kernel* of operator P is a generalized function $k \in \mathcal{D}'(U \times U)$ given by the integral

$$k(x,y) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi} p(x,\xi) f(y)\, d\xi$$

treated in the sense of distributions. If the kernel $k \in C^\infty(U \times U)$ then the associated pseudodifferential operator is called *smoothing* and its degree is set to $-\infty$.

It is convenient to represent the symbols of pseudodifferential operators by asymptotic series. Namely, for any sequence of symbols $\{p_k\}_{k=0}^\infty$, $p_k \in S^{d_k}(U)$, where $\{d_k\}$ is a decreasing sequence of real numbers with $d_k \to -\infty$ for $k \to \infty$, there exists a symbol $p \in S^{d_0}(U)$ such that

$$p - \sum_{k=0}^{n} p_{d_k} \in S^{d_n}(U) \quad \text{for all } n = 0, 1, \dots,$$

and this symbol is uniquely determined modulo the space $S^{-\infty}(U)$ of smoothing symbols. We use the notation: $p \sim \sum_{k=0}^\infty p_{d_k}$.

A standard class of symbols is formed by the so called *classical symbols* for which the exponents $d_k = d - k$, and $p_k(x,\xi)$ are homogeneous functions in ξ of order d_k. The asymptotic decomposition of the symbol in this case takes the form

$$p(x,\xi) \sim \sum_{k=0}^{\infty} p_{d-k}(x,\xi).$$

The leading term $p_d(x,\xi)$ in this decomposition is called the *principal symbol*.

Pseudodifferential operators form an algebra which properties are described in the already mentioned books [4],[11]. We are especially interested in elliptic pseudodifferential operators defined in the following way.

Definition 39. A pseudodifferential operator $P \in \Psi^d(U)$ is called *elliptic* if there exist positive continuous functions c and C in the domain U for which the symbol of operator P satisfies the estimate

$$|p(x,\xi)| \ge c(x)|\xi|^d \quad \text{for } |\xi| \ge C(x), x \in U.$$

Elliptic pseudodifferential operators are invertible modulo smoothing operators, more precisely, we have the following

Proposition 14. *A pseudodifferential operator $P \in \Psi^d(U)$ is elliptic if and only if there exists a symbol $q \in S^{-d}(U)$ such that the corresponding operator $Q \in \Psi^{-d}(U)$ satisfies the relation*

$$P \circ Q = Q \circ P \equiv 1 \ mod \ \Psi^{-\infty}(U).$$

In order to extend the definition of pseudodifferential operators to the operators on manifolds it is necessary to study their behavior under the changes of variables generated by smooth diffeomorphisms. Let $\varphi : U \to V$ be a diffeomorphism of a domain $U \subset \mathbb{R}^n$ onto another domain $V \subset \mathbb{R}^n$. If $P \in \Psi^d(U)$ is a pseudodifferential operator of degree d in the domain U then the fomula

$$\varphi_* P(f) := P(\varphi^* f) \circ \varphi^{-1}$$

determines a pseudodifferential operator in the domain V. In fact, we have the following

Proposition 15. *Suppose that an operator $P \in \Psi^d(U)$ has the following pseudolocality property: both operator P and its adjoint operator $P^\dagger$ map the space $\mathcal{D}(U)$ into itself. Then for a given diffeomorphism $\varphi : U \to V$ the operator $P_\varphi := \varphi_* P$ belongs to $\Psi^d(V)$ and has the same pseudolocality property. Moreover, the kernel p_φ of this operator has the asymptotic decomposition of the form*

$$p_\varphi(x,\xi) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} q_\alpha(x,\xi) D_\xi^\alpha(\psi(x), {}^t(\psi'(x)^{-1})\xi)$$

where $\psi := \varphi^{-1}$, $q_0(x,\xi) = 1$ and $q_\alpha(x,\xi)$ is a polynomial in ξ of degree$\leq \frac{1}{2}|\alpha|$.

Explicit expressions for the coefficients $q_\alpha(x,\xi)$ may be found in the book [4] (Volume III, Theorem 18.1.17), we note only that for the principal symbol $p_{\varphi,d}(x,\xi)$ the change of variables formula has the form

$$p_{\varphi,d}(\varphi(x),\xi) = p_d(x, {}^t\varphi'(x)\xi).$$

With this proposition we can define pseudodifferential operators on a compact manifold in such a way that they will satisfy automatically the pseudolocality property from the last proposition.

Definition 40. Let M be a compact manifold. A linear operator $P : \mathcal{D}(M) \to C^\infty(M)$ is called the *pseudodifferential operator* of degree d if its kernel is smooth away from the diagonal in $M \times M$ and for any coordinate chart (U,φ) the operator $\varphi_* P$, acting from $\mathcal{D}(\varphi(U))$ to $C^\infty(\varphi(U))$, is a pseudodifferential operator from the space $\Psi^d(\varphi(U))$. Such operator is called the *classical pseudodifferential operator* if all its local expressions are pseudodifferential operators with classical symbols.

The above change of variables formula for the principal symbol implies that this symbol is correctly defined as a function on the cotangent bundle $T^*M \to M$. Elliptic pseudodifferential operators are defined as the operators for which all their local expressions are elliptic operators.

2.1.4 Wodzicki residue

Prior the definition of Wodzicki residue we recall several auxiliary facts on homogeneous functions and forms on the cotangent bundles of smooth manifolds.

Let M be a smooth compact manifold of dimension $n > 1$ and T^*M is its cotangent bundle with local coordinates given by (x, ξ) where $x = (x_1, \ldots, x_n)$ are local coordinates on M and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ are coordinates in the fibre T_x^*M. Denote by σ_ξ the differential $(n-1)$-form on $\mathbb{R}^n \setminus 0$ defined by

$$\sigma_\xi := \sum_{j=1}^{n} (-1)^{j-1} \xi_j d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n$$

where the "hat" over $d\xi_j$ means that this term should be omitted. The form σ_ξ coincides with the inner product of the volume form $d^n\xi = d\xi_1 \wedge \ldots \wedge d\xi_n$ and Euler vector field $E = \sum_{j=1}^{n} \xi_j \partial/\partial\xi_j$: $\sigma_\xi = E \lrcorner\, d^n\xi$.

Lemma 11. *For any homogeneous function $p_{-n}(\xi)$ of homogeneity degree $-n$ the form $p_{-n}\sigma_\xi$ on the space $\mathbb{R}^n \setminus 0$ is closed.*

Proof. Indeed,

$$dp_{-n} \wedge \sigma_\xi = dp_{-n} \wedge (E \lrcorner\, d^n\xi) = E \lrcorner\, dp_{-n} \wedge d^n\xi = -np_{-n} d^n\xi$$

which implies that

$$d(p_{-n}\sigma_\xi) = dp_{-n} \wedge \sigma_\xi + p_{-n} d\sigma_\xi = -np_{-n}d^n\xi + np_{-n}d^n\xi = 0.$$

□

We shall use the above lemma to compute the integrals of the form $\int_{S^{n-1}} p_{-n}|\sigma_\xi|$. According to this lemma, we can replace the integration space given by the unit sphere S^{n-1} by any section of the bundle $\mathbb{R}^n \setminus 0 \to S^{n-1}$, the result will be the same.

Recall that according to Euler theorem, the homogeneous functions f of homogeneity degree λ satisfy the identity

$$\frac{1}{n+\lambda} \sum_{j=1}^{n} \frac{\partial(\xi_j f)}{\partial \xi_j} = \frac{1}{n+\lambda}(nf + Ef) = f.$$

For $\lambda = -n$ this identity has no sense, however the following assertion holds.

Lemma 12. *The integral*

$$\int_{S^{n-1}} p_{-n}|\sigma_\xi| = 0$$

if and only if the function p_{-n} is the sum of derivatives.

Proof. It is well known that the kernel of Laplace operator acting on the smooth functions on a compact manifolds, consists only of the constants. If the considered integral vanishes then the right hand side of the equation

$$\Delta_{S^{n-1}} f = p_{-n}|S^{n-1}$$

is orthogonal to the kernel of $\Delta_{S^{n-1}}$ so this equation should have a solution. We extend this solution f to a function $\tilde{f}$, defined on the whole space $\mathbb{R}^n \setminus 0$, by setting: $\tilde{f}(\xi) := |\xi|^{-(n-2)} f(\xi/|\xi|)$. The Laplace equation for this function will rewrite as

$$\Delta_{\mathbb{R}^n} \tilde{f} = |\xi|^{-n} p_{-n}(\xi/|\xi|) = p_{-n}(\xi).$$

Conversely, assume first that p_{-n} is the derivative of some function, say, $p_{-n} = \partial q_{1-n}/\partial \xi_1$ where q_{1-n} is some homogeneous function of degree $1-n$. The cycle S^{n-1} may be replaced by the cycle of the form $S^{n-2} \times \mathbb{R}$ since the function q_{1-n}, taking into account its homogeneity degree, should tend to zero for $|\xi_1| \to \infty$. Denoting $(\xi_2, \ldots, \xi_n)$ by η, we get

$$\int_{S^{n-1}} p_{-n} |\sigma_\xi| = \pm \int_{S^{n-2}} \int_{-\infty}^{\infty} \frac{\partial q_{1-n}}{\partial \xi_1} |d\xi_1| |\sigma_\eta| = 0.$$

The same argument goes on in the case when p_{-n} is the sum of derivatives with respect to different variables. □

We shall also need the notion of density on a manifold. Let us start from the case of the real vector space V of dimension n.

Definition 41. A *density* on the vector space V is the continuous map $\lambda : V^n \to \mathbb{R}$ having the following property

$$\lambda(Av_1, \ldots, Av_n) = |\det A| \lambda(v_1, \ldots, v_n)$$

for all $v_1, \ldots, v_n \in V$, $A \in \operatorname{End} V$.

If ω is the volume form on V then it determines a density $|\omega|$ on V by the formula: $|\omega|(v_1, \ldots, v_n) := |\omega(v_1, \ldots, v_n)|$.

The definition of densities is easily extended to arbitrary Riemannian manifolds by the following

Lemma 13. *Let (M, g) be a Riemannian manifold with Riemannian metric g. Then there exists a unique density $|\nu_g|$ on M which takes values 1 on all orthonormal bases of tangent spaces T_xM, $x \in M$. If the vectors $v_1, \ldots, v_n \in T_xM$ then*

$$|\nu_g|(v_1, \ldots, v_n) = |\det(g_x(v_i, v_j))|^{1/2}.$$

It is natural to call this density Riemannian.

The existence of the Riemannian density follows from the fact that the bundle of densities on any smooth manifold M (with the transition functions given by the modulus of Jacobians) is trivial since there are no objections to the construction of a non-zero section (in contrast with the bundle of volume forms which is trivial only in the case of oriented manifolds). In Riemannian case we can, using the Riemannian density, identify the space of all densities on the manifold M with the direct product $M \times \mathbb{R}$. The local expression for the Riemannian density is written in the form

$$\sqrt{\det g} |dx_1 \wedge \ldots \wedge dx_n| = \sqrt{\det g} |d^n x|.$$

It follows from the change of variables formula on M that there exists a linear form $\int_M$ on the space of sections of the bundle of densities, defined in a unique way, which invariant under diffeomorphisms and coincides with the Lebesgue integral in local charts.

We can give now the definition of the Wodzicki residue. Recall that the classical pseudodifferential operator on a compact manifold M is a linear operator P with the symbol having local expressions of the form

$$p(x,\xi) \sim \sum_{k=0}^{\infty} p_{d-k}(x,\xi)$$

where the function $p_{d-k}(x,\xi)$ is homogeneous in ξ of degree $d-k$.

Theorem 14 (Wodzicki)**.** *Suppose that P is a classical pseudodifferential operator defined on a smooth compact manifold M of dimension n. Then there exists a density $res_x P$ on M having local expressions of the form*

$$res_x P = \left(\int_{|\xi|=1} p_{-n}(x,\xi)|\sigma_\xi| \right) |d^n x|. \tag{2.4}$$

The integral determined by this density is called the Wodzicki residue of operator P:

$$Res\, P := \int_M res_x P. \tag{2.5}$$

Proof. The change of variables formula for pseudodifferential operators gives a formula for the transformation of a pseudodifferential operator $P \in \Psi^d(U)$ to the pseudodifferential operator $\varphi_* P \in \Psi^d(V)$ under the action of a diffeomorphism $\varphi : U \to V$ from a domain $U \subset \mathbb{R}^n$ onto a domain $V \subset \mathbb{R}^n$. The symbol $p(x,\xi)$ of operator P under this map is transformed to the symbol $p_\varphi(x,\xi) =: \tilde{p}(x,\xi)$ of operator $\varphi_* P$ given by the formula (where $\xi = {}^t\psi'(x)\eta$):

$$\tilde{p}(x, {}^t\psi'(x)\eta) = \sum_\alpha c_\alpha(x,\eta)\partial_\eta^\alpha p(\psi(x),\eta).$$

Here $c_0(x,\eta) = 1$ and the other coefficients $c_\alpha(x,\eta)$ are polynomials in η. It means, in particular, that the coefficient $\tilde{p}_{-n}(x, {}^t\psi'(x)\eta)$ differs from the coefficient $p_{-n}(\psi(x),\eta)$ by the sum of the terms which are derivatives in ξ.

Let us see how the integral $\int_{|\xi|=1} p_{-n}(x,\xi)|\sigma_\xi|$ changes under linear changes of the variable ξ for a fixed x. Suppose that such change is given by a map h. It is easy to see that $h^*\sigma_\xi = (\det h)\sigma_{h\xi}$.

Note that the integral over the sphere $S = \{\xi : \ |\xi| = 1\}$ of the form $p_{-n}|\sigma_\xi|$ coincides (up to sign) with the integral of this form over the image $h(S)$:

$$\int_{|\xi|=1} p_{-n}(x,\xi)|\sigma_\xi| = \pm \int_{h(S)} p_{-n}(x,\xi)$$

since $h(S)$ is homologous to S (the sign "plus" corresponds to the case when h preserves the orientation while "minus" corresponds to the opposite case). Hence

$$\int_S p_{-n}(x,\xi)|\sigma_\xi| = \pm \int_S h^* \left(p_{-n}(x,\xi)|\sigma_\xi|\right) = |\det h| \int_S p_{-n}(x,h\xi)|\sigma_{h\xi}|.$$

Setting $y := \psi(x)$, $\xi = {}^t\psi'(x)\eta$, we obtain the following formula for the transformed residue:

$$\int_{|\xi|=1} \tilde{p}(x,\xi)|\sigma_\xi||d^n x| = |\det \psi'(x)| \int_{|\eta|=1} \tilde{p}_{-n}(x, {}^t\psi'(x)\eta)|\sigma_\eta||d^n x| =$$
$$\int_{|\eta|=1} \tilde{p}_{-n}(x, {}^t\psi'(x)\eta)|\sigma_\eta||d^n y| = \int_{|\eta|=1} p_{-n}(y,\eta)|\sigma_\eta||d^n y|$$

where we have used the fact that the cycles $|\xi| = 1$ and $|\eta| = 1$ are homologous to each other for a fixed x and the terms, consisting of derivatives, do not contribute to the last integral. Then the latter formula implies that the density $\mathrm{res}_x P$ is correctly defined so the integral determined by this density does not depend on choice of local coordinates. □

Denote by $\mathcal{P}(M)$ the quotient of the algebra of classical pseudodifferential operators on the manifold M with respect to the ideal of smoothing classical operators. The Wodzicki residue determines a trace on the algebra $\mathcal{P}(M)$ and (for $n > 1$) this trace is uniquely defined (up to multiplication by a non-zero constant). This is asserted by another Wodzicki theorem the proof of which can be found, for example, in [3], Theorem 7.6. It suggests the idea that Wodzicki residue should be related with the Dixmier trace introduced above. We shall return to this question in the next section.

Let us give an example of computation of Wodzicki residue for the Laplace operator on a compact Riemannian manifold.

Example 8. Let $P = \Delta$ be the Laplace–Beltrami operator on a compact Riemannian n-dimensional manifold (M, g). Then

$$\mathrm{Res}\,\Delta^{-n/2} = \Omega_n$$

where $\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of the sphere S^{n-1}.

Indeed, since Δ is an operator of the second order the operator $\Delta^{-n/2}$ has the order $-n$ and the principal symbol of this operator has the form $(g^{ij}(x)\xi_i\xi_j)^{-n/2}$ where (g_{ij}) is the metric tensor of the manifold M, (g^{ij}) is the inverse matrix of (g_{ij}). After the change of variables $y = \psi(x)$, $\xi = {}^t\psi'(x)\eta$, for which $\psi'(x) = (\det g)^{1/2}$, the principal symbol will transform to $|\eta|^{-n}$, and the residue density will be equal to

$$\mathrm{res}_x P = \Omega_n |d^n y| = \Omega_n \det \psi'(x)|d^n x| = \Omega_n |\nu_g|$$

where $|\nu_g|$ is the Riemannian density. It follows that $\mathrm{Res}\,\Delta^{-n/2} = \Omega_n$.

2.1.5 Connes trace theorem

We have pointed out already that the Wodzicki residue and Dixmier trace of pseudodifferential operators should be related with each other. The concrete expression for this relation is given by the Connes trace theorem.

Theorem 15. *Let P be an elliptic pseudodifferential operator of degree $-n$ on a compact Riemannian manifold (M, g). Then the operator P belongs to the space*

$\mathcal{L}^{1,\infty}$ and is measurable. Moreover, its Dixmier trace is related to the Wodzicki residue by the formula

$$Tr^{+}\,P = \frac{1}{n(2\pi)^n} Res\,P.$$

The proof of this theorem, which is omitted here, may be found in the original book [2] and [3], Theorem 7.18.

The Connes theorem implies the following

Corollary 6. *For an arbitrary smooth function $a \in C^{\infty}(M)$ the following equality*

$$\int_M a(x)|\nu_g| = \frac{n(2\pi)^n}{\Omega_n} Tr^{+}(a\Delta_g^{-n/2})$$

holds.

Proof. The operator $a\Delta^{-n/2}$ is a pseudodifferential operator of degree $-n$ with the principal symbol $a_{-n}(x,\xi) := a(x)(g^{ij}\xi_i\xi_j)^{-n/2}$ so (cf. Example 8) the density of Wodzicki residue for this operator has the form

$$\mathrm{res}_x(a\Delta^{-n/2}) = \Omega_n a(x)|\nu_g|.$$

Hence, the left hand side of the required equality coincides with $\Omega_n^{-1}\mathrm{Res}(a\Delta^{-n/2})$. Now the assertion of the corollary follows from the trace theorem. $\square$

2.2 Noncommutative differential calculus

2.2.1 Universal differential algebra

Let $\mathcal{E}$ be a bimodule over a unital algebra A.

Definition 42. A *derivation* of the algebra A with values in the bimodule $\mathcal{E}$ is a linear map $D : A \to \mathcal{E}$ satisfying the Leibniz rule

$$D(ab) = (Da)b + a(Db).$$

This definition immediately implies that $D(1_A) = 0$ since $D(1_A) = 2D(1_A)$.

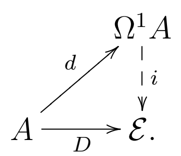
Denote by $\mathrm{Der}(A, \mathcal{E})$ the set of all derivations of the algebra A with values in $\mathcal{E}$. Any element $s \in \mathcal{E}$ determines a derivation from $\mathrm{Der}(A, \mathcal{E})$ by the formula

$$(\mathrm{ad}\, s)a := sa - as.$$

Such derivation is called *inner*.

The set $\mathrm{Der}(A) \equiv \mathrm{Der}(A, A)$ of derivations of the algebra A is a Lie algebra since the commutator of two derivations is again a derivation.

Our next goal is to construct a bimodule $\Omega^1 A$ given together with a derivation $d : A \to \Omega^1 A$ which has the following universal property: for any derivation D of the algebra A with values in the bimodule $\mathcal{E}$ there exists a unique bimodule morphism $i : \Omega^1 A \to \mathcal{E}$ making the following diagram commutative:



In other words, the linear map

$$\mathrm{Hom}_A(\Omega^1 A, \mathcal{E}) \longrightarrow \mathrm{Der}(A, \mathcal{E})$$

given by the formula $\varphi \longmapsto \varphi \circ d$ should be an isomorphism.

Construction of the bimodule $\Omega^1 A$

Let $A \otimes A$ be the tensor product of the algebra A with itself, considered as an A-bimodule provided with the action of elements of A given on pure tensors by the formulas:

$$a(b \otimes c) \equiv (a \otimes 1_A)(b \otimes c) = (ab) \otimes c,$$
$$(a \otimes b)c \equiv (a \otimes b)(1_A \otimes c) = a \otimes (bc).$$

Define a derivation $d : A \to A \otimes A$ of the algebra A with values in $A \otimes A$ by the formula:

$$da := 1_A \otimes a - a \otimes 1_A$$

(later on we shall omit the lower index A in the notation 1_A when it does not lead to misunderstanding). Then

$$d(ab) = 1 \otimes (ab) - (ab) \otimes 1 = a \otimes b - (ab) \otimes 1 + 1 \otimes (ab) - a \otimes b = adb + (da)b$$

since

$$adb = a(1 \otimes b) - a(b \otimes 1) = (a \otimes 1)(1 \otimes b) - (a \otimes 1)(b \otimes 1) = a \otimes b - (ab) \otimes 1,$$
$$(da)b = (1 \otimes a)b - (a \otimes 1)b = (1 \otimes a)(1 \otimes b) - (a \otimes 1)(1 \otimes b) = 1 \otimes (ab) - a \otimes b.$$

So d is indeed a derivation of the algebra A with values in $A \otimes A$.

Denote by $\Omega^1 A$ the submodule in $A \otimes A$ generated by elements of the form adb. It coincides with the kernel of the map

$$m : A \otimes A \longrightarrow A, \quad a \otimes b \longmapsto ab.$$

Indeed, if an element $\sum_k a_k \otimes b_k \in A \otimes A$ belongs to $\mathrm{Ker}\, m$, i.e. $\sum_k a_k b_k = 0$, then

$$\sum_k a_k \otimes b_k = \sum_k a_k(1 \otimes b_k - b_k \otimes 1) = \sum_k a_k db_k$$

which implies the required assertion.

Introduce the structure of an A-bimodule on $\Omega^1 A$ by setting

$$a(bdc) := (ab)dc, \quad (adb)c := ad(bc) - (ab)dc.$$

The constructed bimodule is called the *bimodule of universal 1-forms over the algebra A*.

Suppose now that $\mathcal{E}$ be an arbitrary bimodule over the algebra A and $D : A \to \mathcal{E}$ is a derivation of A with values in $\mathcal{E}$. Define the map $i : \Omega^1 A \to \mathcal{E}$ by setting it equal on pure tensors from $A \otimes A$ to

$$i(a \otimes b) := a(Db)$$

and restricting then to $\Omega^1 A \subset A \otimes A$. This map is an A-bimodule morphism which implies that the bimodule $\Omega^1 A$ indeed has the universal property formulated in the beginning of this section.

Differential graded algebra

Definition 43. A *differential graded algebra* (briefly: DG-*algebra*) $(R^\bullet, \delta)$ is an associative algebra

$$R^\bullet = \bigoplus_{n=0}^{\infty} R^n$$

which is provided with a *graded product*, i.e. the product having the property $R^m \cdot R^n \subseteqq R^{m+n}$, and a *differential* δ, i.e. a linear map satisfying the conditions:

1. δ is a map of degree $+1$, i.e. it sends $R^n \to R^{n+1}$,
2. $\delta^2 = 0$,
3. δ is an *odd derivation*, i.e. it satisfies the Leibniz rule of the form

$$\delta(\omega^n \eta) = (\delta\omega^n)\eta + (-1)^n \omega^n \delta\eta$$

where $\omega^n \in R^n$.

Our goal is to construct a DG-algebra

$$\Omega^\bullet A = \bigoplus_{n=0}^{\infty} \Omega^n A$$

with differential d for which the two first summands have the form: $\Omega^0 A = A$, $\Omega^1 A$ defined above, and the differential d extends the constructed derivation from the algebra A to $\Omega^1 A$. Moreover, we would like to have the DG-algebra havinf the following universal property: if $(R^\bullet, \delta)$ is another DG-algebra then any algebra homomorphism $\psi : A \to R^0$ should extend to an algebra homomorphism $\psi : \Omega^\bullet A \to R^\bullet$ of degree zero intertwining the differentials d and δ.

Denote by $\bar{A}$ the quotient algebra $\bar{A} := A/\mathbb{C}$ and by $\bar{a}$ the image of an element $a \in A$ under the projection to $\bar{A}$. The introduced bimodule $\Omega^1 A$ may be identified with

$$\Omega^1 A \cong A \otimes \bar{A}$$

by the map: $a \otimes \bar{b} \mapsto adb$. This identification is correctly defined since $d(1_A) = 0$. If we introduce in $A \otimes \bar{A}$ the left and right multiplication by elements $c \in A$ by the formulas

$$\begin{aligned} c(a_0 \otimes \bar{a}_1) &= (ca_0) \otimes \bar{a}_1, \\ (a_0 \otimes \bar{a}_1)c &= a_0 \otimes \bar{a}_1 c - (a_0 a_1) \otimes \bar{c} \end{aligned}$$

then the map $A \otimes \bar{A} \to \Omega^1 A$ becomes an isomorphism since

$$\begin{aligned} &c(a_0 \otimes \bar{a}_1) = (ca_0) \otimes \bar{a}_1 \longmapsto (ca_0) \otimes da_1, \\ &(a_0 \otimes \bar{a}_1)c = a_0 \otimes \bar{a}_1 c - (a_0 a_1) \otimes \bar{c} \longmapsto a_0 \otimes da_1 c - (a_0 a_1) \otimes dc = (a_0 da_1)c. \end{aligned}$$

We set now by definition

$$\Omega^2 A := \Omega^1 A \otimes_A \Omega^1 A = (A \otimes \bar{A}) \otimes_A (A \otimes \bar{A}) = A \otimes \bar{A} \otimes \bar{A}.$$

More generally, define

$$\Omega^n A := \Omega^1 A \otimes_A \ldots \otimes_A \Omega^1 A \ (n \text{ times})$$

so that

$$\Omega^n A = A \otimes \bar{A}^{\otimes n}.$$

The differential $d : A \otimes \bar{A}^{\otimes n} \to A \otimes \bar{A}^{\otimes (n+1)}$ is given by the shift

$$d(a_0 \otimes \bar{a}_1 \otimes \ldots \otimes \bar{a}_n) := 1_A \otimes \bar{a}_0 \otimes \bar{a}_1 \otimes \ldots \otimes \bar{a}_n.$$

Then $d^2 = 0$ since $\bar{1}_A = 0$ in the algebra $\bar{A}$.

Identifying, as before, $A \otimes \bar{A}^{\otimes n}$ with $(\Omega^1 A)^{\otimes n}$ we shall have

$$a_0 \otimes \bar{a}_1 \otimes \ldots \otimes \bar{a}_n = a_0 da_1 \ldots da_n.$$

Introduce on $\Omega^\bullet A$ the structure of an A-bimodule. The left multiplication is given by the evident formula:

$$c(a_0 da_1 \ldots da_n) = (ca_0) da_1 \ldots da_n.$$

In order to define the right multiplication we use the Leibniz rule: $da \cdot b = d(ab) - adb$. Then

$$\begin{aligned}(a_0 da_1 \ldots da_n)c &= a_0 da_1 \ldots da_{n-1} d(a_n c) - a_0 da_1 \ldots da_{n-1} a_n dc = \ldots \\ &= (-1)^n (a_0 a_1) da_2 \ldots da_n c + \sum_{j=1}^{n-1} (-1)^{n-j} a_0 a_1 \ldots d(a_i a_{j+1}) \ldots da_n dc + \\ &\qquad + a_0 da_1 \ldots da_{n-1} d(a_n c).\end{aligned}$$

At last, we define the product in $\Omega^\bullet A$ by setting:

$$(a_0 da_1 \ldots da_n)(b_0 db_1 \ldots db_m) := (a_0 da_1 \ldots da_n \cdot b_0)\, db_1 \ldots db_m.$$

Thus, $\Omega^\bullet A$ becomes a DG-algebra called the *universal DG-algebra* over the algebra A.

We have the following useful formulas:

$$d(a_0 da_1 \ldots da_n) = 1_A da_0 da_1 \ldots da_n = da_0 da_1 \ldots da_n$$

and

$$a_0 [d, a_1] \ldots [d, a_n] \cdot 1_A = a_0 da_1 \ldots da_n.$$

The first of them rephrases the definition of differential using the identification of $A \otimes \bar{A}^{\otimes n}$ with $(\Omega^1 A)^{\otimes n}$, and for the proof of the one we note that

$$\begin{aligned}[][d, a_n] \cdot 1_A &= da_n - a_n d1_A = da_n, \\ [d, a_{n-1}] da_n &= d(a_{n-1} da_n) = da_{n-1} da_n\end{aligned}$$

and so on by induction.

We check now the universal property for the constructed DG-algebra $\Omega^\bullet A$. Suppose that we are given another DG-algebra $(R^\bullet, \delta)$ and an algebra homomorphism $\psi : A \to R^0$. Then its extension to a morphism $\psi : \Omega^\bullet A \to R^\bullet$ is given by the formula

$$\psi(a_0 da_1 \ldots da_n) := \psi(a_0) \delta\left(\psi(a_1)\right) \ldots \delta\left(\psi(a_n)\right).$$

2.2.2 Cycles and Fredholm modules

Cycles

Definition 44. A *cycle* of dimension n is the DG-algebra

$$\Omega^\bullet = \bigoplus_{k=0}^{n} \Omega^k$$

given together with the *integral* $\int$, i.e. a linear map $\int : \Omega^\bullet \to \mathbb{C}$ such that:

1. $\int \omega^k = 0 \quad \text{for } k < n;$
2. $\int d\omega^{n-1} = 0;$
3. $\int \omega^k \omega^l = (-1)^{kl} \int \omega^l \omega^k.$

A *cycle over an algebra* A is a cycle $(\Omega^\bullet, d, \int)$ together with a homomorphism $A \to \Omega^0$.

The standard examples of cycles of dimension n are given by the de Rham complex over an n-dimensional smooth compact manifold and the algebra of smooth matrix functions on $\mathbb{R}^n$ with appropriate growth condition at infinity where the integral is given by $\int \omega^n := \int \operatorname{tr} \omega^n$. Less trivial examples are related to Fredholm modules which we are going to define.

Fredholm modules and cycles defined by such modules

Definition 45. An *odd Fredholm module* over a C^*-algebra A is an involutive representation σ of this algebra A in a Hilbert space $\mathcal{H}$ provided with a *symmetry operator*, i.e. a linear operator S such that $S = S^*$ and $S^2 = I$, satisfying the condition

$$[S, \sigma(a)] \in \mathcal{K}(\mathcal{H}) \quad \text{for all } a \in A.$$

An *even Fredholm module* is given by a representation $\sigma = \sigma^0 \oplus \sigma^1$ of the algebra A in a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ with an odd symmetry operator S satisfying the same conditions as in the odd case.

A Fredholm module $(A, \mathcal{H}, S)$ generates a cycle over the algebra A with $\Omega^0 = A$ while the symmetry operator S determines a $\mathbb{Z}_2$-grading on the algebra of bounded linear operators $\mathcal{L}(\mathcal{H})$. Indeed, we can write down an arbitrary linear operator T in the form

$$T = T_+ + T_- \quad \text{where } T_\pm = \frac{T \pm STS}{2}.$$

Then

$$(TR)_+ = T_+R_+ + T_-R_- \quad \text{and} \quad (TR)_- = T_+R_- + T_-R_+.$$

Moreover,

$$T = T_+ \iff T = STS \quad \text{and} \quad T = T_- \iff T + STS = 0.$$

In order to define the integral we have to impose on the considered Fredholm modules an additional condition of *summability* of order n:

$$[S, \sigma(a)] \in \mathcal{L}^{n+1}(\mathcal{H}),$$

where the number n is assumed to be odd for odd Fredholm modules and even for even Fredholm modules.

Differentials

Assuming that the summability condition is satisfied we define the *differential* by setting:

$$da = i[S, \sigma(a)] = 2iS\sigma(a)_-$$

for $a \in A$. We shall omit further on the symbol "σ" so that the last formula will be written in the form

$$da = i[S, a] = 2iSa_-.$$

In other words, the differential d chooses the S-odd part of the element a and the summability condition may be rewritten in the form $da \in \mathcal{L}^{n+1}$.

The multiplier i is introduced in order to ensure that the differential d commutes with the involution:

$$d(a^*) = (da)^\dagger$$

where in the right hand side we use the Hermitian conjugation.

Having defined the differential d, we can also introduce the *differentials of higher orders*. For that consider the space of 1-forms on the algebra of bounded linear operators $\mathcal{L}(\mathcal{H})$. By these we mean the operators of the form $a_0 da_1$, where $a_0, a_1 \in A$, and their linear combinations. The differential of the second order on 1-forms is defined by the formula

$$d(a_0 da_1) = i[S, a_0 da_1] = i[S, a_0] i[S, a_1] = da_0 da_1$$

(check that the second equality indeed holds!). Note that in this formula, as also in the next ones, we mean by the commutator the supercommutator so that the commutator $[S, a_0 da_1]$ is in fact the anti-commutator since $a_0 da_1$ is a 1-form.

From this definition it follows that

$$d(a_0 da_1) = 2iS(a_0 da_1)_+,$$

i.e. the differential of the 2nd order, opposite to the differential of the 1st order, chooses the S-even part of the form $a_0 da_1$, belonging to $\mathcal{L}^{n+1} \cdot \mathcal{L}^{n+1} \subset \mathcal{L}^{(n+1)/2}$, while the S-odd part of the form $a_0 da_1$, equal to $(a_0)_+(da_1)_- = (a_0)_+ da_1$ belongs to $\mathcal{L}^{n+1}$. (Further on we omit the prefix S speaking on the oddness or evenness of the forms.)

Consider next the space of 2-forms generated by the elements $a_0 da_1 da_2$ where $a_0, a_1, a_2 \in A$. The differential of the 3rd order will choose the odd part of $a_0 da_1 da_2$, belonging to $\mathcal{L}^{(n+1)/3}$, while the even part of $a_0 da_1 da_2$ will belong to $\mathcal{L}^{(n+1)/2}$.

In the general case we consider the space Ω^k of k-forms generated by the operators

$$a = a_0 da_1 \dots da_k \quad \text{with } a_0, a_1, \dots, a_k \in A.$$

If $k = 2r$, i.e. $a \in \Omega^{2r}$, then $a_+ \in \mathcal{L}^{(n+1)/2r}$, $a_- \in \mathcal{L}^{(n+1)/(2r+1)}$. In the case when $k = 2r - 1$, i.e. $a \in \Omega^{2r-1}$, we shall have: $a_+ \in \mathcal{L}^{(n+1)/2r}$, $a_- \in \mathcal{L}^{(n+1)/(2r-1)}$.

The product of forms coincides with their composition, and the differential is given by the formula

$$d(a_0 da_1 \dots da_k) = i[S, a_0 da_1 \dots da_k] = i[S, a_0[iS, a_1] \dots i[S, a_k]] = da_0 da_1 \dots da_k.$$

This implies the general formula

$$d\omega = i[S, \omega] \quad \text{for } \omega \in \Omega^\bullet.$$

We use now the *polarization* of the Hilbert space $\mathcal{H}$ generated by the symmetry operator S:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

where $\mathcal{H}_\pm$ is the (± 1)-eigenspace of operator S. We shall write down linear operators acting in $\mathcal{H}$ in the block form so that

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{pmatrix}.$$

Introduce also the following notations:

$$a_+ = \begin{pmatrix} a_{++} & 0 \\ 0 & a_{--} \end{pmatrix} \quad \text{and} \quad a_- = \begin{pmatrix} 0 & a_{+-} \\ a_{-+} & 0 \end{pmatrix}.$$

The formula for the differential of the 1st order may be rewritten in the form

$$da = i[S, a] = 2i \begin{pmatrix} 0 & a_{+-} \\ -a_{-+} & 0 \end{pmatrix}.$$

Integral

We turn now to the definition of the integral. Introduce first of all the *conditional trace* of operator $T \in \mathcal{L}(\mathcal{H})$ by setting

$$\mathrm{Tr}' T := \mathrm{Tr}\, T_+.$$

Note that $\mathrm{Tr}' T = \mathrm{Tr}\, T$ if $T \in \mathcal{L}^1$ due to the cyclicity of the usual trace.

Assume first that n is odd. Then $(\omega^n)_+ \in \mathcal{L}^{1+1/n} \subset \mathcal{L}^1$ so the integral

$$\int \omega^n := \mathrm{Tr}' \omega^n = -\frac{i}{2} \mathrm{Tr}(S d\omega^n)$$

is well defined. The second equality in this formula is implied by the following chain of equalities

$$S d\omega^n = iS[S, \omega^n] = iS(S\omega^n + \omega^n S) = i(\omega^n + S\omega^n S) = 2i(\omega^n)_+.$$

For the forms ω^k with $k < n$ we set $\int \omega^k = 0$ by definition.

Let us show that the constructed integral has the properties listed in Definition 44. First of all

$$\int d\omega^{n-1} = -\frac{i}{2} \mathrm{Tr}(S d^2 \omega^{n-1}) = 0.$$

Secondly, consider the forms ω^k, ω^l with $k+l=n$. Assume for definiteness that k is odd and l is even. Then

$$\int \omega^k\omega^l = -\frac{i}{2}\mathrm{Tr}(Sd(\omega^k\omega^l)) =$$
$$= -\frac{i}{2}\mathrm{Tr}(Sd\omega^k\omega^l - S\omega^k d\omega^l) = -\frac{i}{2}\mathrm{Tr}(-d\omega^l S\omega^k - \omega^l Sd\omega^k) =$$
$$= -\frac{i}{2}\mathrm{Tr}(Sd\omega^l\omega^k + S\omega^l d\omega^k) = -\frac{i}{2}\mathrm{Tr}(Sd(\omega^l\omega^k)) = \int \omega^l\omega^k.$$

In the third equality we have used the cyclicity property: $\mathrm{Tr}(TR) = \mathrm{Tr}(RT)$ for operators $T \in \mathcal{L}^p$, $R \in \mathcal{L}^q$ with $1/p + 1/q = 1$. In our case $p = k/(n+1)$, $q = (l+1)/(n+1)$.

Assume now that n is even and again $k+l=n$. Then $(\omega^n)_- \in \mathcal{L}^1$. Denote by χ the grading operator on $\mathcal{H}$ having the (± 1)-eigenspaces coinciding respectively with $\mathcal{H}^0$ and $\mathcal{H}^1$. In this case we define the integral as

$$\int \omega^n := \mathrm{Tr}'(\chi\omega^n) = -\frac{i}{2}\mathrm{Tr}(\chi Sd\omega^n)$$

and set: $\int \omega^k = 0$ for forms ω^k with $k < n$. The property of closedness is again evident:

$$\int d\omega^{n-1} = -\frac{i}{2}\mathrm{Tr}(\chi Sd^2\omega^{n-1}) = 0,$$

and the permutation property

$$\int \omega^k\omega^l = (-1)^{kl}\int \omega^l\omega^k$$

is checked as above taking into account the equality: $\chi\omega^k = (-1)^k\omega^k\chi$.

Example 9 (Hilbert transform)**.** The *Hilbert transform* of a function $h \in L^2(\mathbb{R})$, defined on the real line, is the integral of the form

$$Sh(x) := \frac{i}{\pi}\lim_{\varepsilon\to 0+}\int_{|t|>\varepsilon}\frac{h(x-t)}{t}\,dt =: \frac{i}{\pi}P.V.\int_{\mathbb{R}}\frac{h(x-t)}{t}\,dt.$$

The Fourier transform of this function coincides with

$$\mathcal{F}(Sh)(\xi) = (\operatorname{sgn}\xi)\mathcal{F}h(\xi).$$

The Hilbert transform is the only linear bounded operator in $L^2(\mathbb{R})$ which commutes with translations and dilatations.

Example 10 (Riesz operators)**.** The Riesz operators are the multi-dimensional analogues of the Hilbert transform. The *Riesz operators* R_j, $1 \le j \le n$, act in $L^2(\mathbb{R}^n)$ by the formula

$$R_jh(x) := \frac{2i}{\Omega_{n+1}}P.V.\int_{\mathbb{R}^n}\frac{t_jh(x-t)}{|t|^{n+1}}\,dt$$

where Ω_{n+1} is the volume of the unit sphere S^n equal to

$$\Omega_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}\,.$$

The Fourier transform of this function is equal to

$$\mathcal{F}(R_j h)(\xi) = \frac{\xi_j}{|\xi|}\mathcal{F}h(\xi).$$

The Riesz operators also commute with translations and dilatations, moreover

$$\sum_{j=1}^{n} R_j^2 = 1.$$

The Fredholm module, associated with Riesz operators, is constructed from the collection of $(N \times N)$-matrices $\gamma_1, \ldots, \gamma_n$ such that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

These matrices γ_j coincide with the Dirac matrices generating the spin representation of the Clifford algebra $\mathrm{Cl}^{\mathbb{C}}(\mathbb{R}^n)$ in the space $\mathbb{C}^N$ where $N = 2^{[n/2]}$. Having such collection of matrices γ_j, we can define the symmetry operator by the formula

$$S := \sum_{j=1}^{n} \gamma_j R_j.$$

The matrices γ_j can be constructed in the following way. For $n = 1$ we set $\gamma_1^{(1)} = 1$ and for all odd $n > 1$ define the collections of functions $\gamma_j^{(n)}$ by induction setting:

$$\gamma_j^{(n)} := \begin{pmatrix} 0 & \gamma_j^{(n-2)} \\ \gamma_j^{(n-2)} & 0 \end{pmatrix} \quad \text{for } j = 1, \ldots, n-2,$$

and

$$\gamma_{n-1}^{(n)} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_n^{(n)} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, for $n = 3$ we get the Pauli matrices. For even n we set: $\gamma_j^{(n)} := \gamma_j^{(n+1)}$, $j = 1, \ldots, n$.

2.2.3 Connections

Definition and existence of connections

Definition 46. Let $\mathcal{E}$ be a right A-module over an algebra A. Consider the right A-module $\mathcal{E} \otimes_A \Omega^1 A$. A *connection* on $\mathcal{E}$ is a linear map

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_A \Omega^1 A$$

satisfying the Leibniz rule:

$$\nabla(sa) = (\nabla s)a + s \otimes da$$

where $s \in \mathcal{E}$, $a \in A$.

The operator, determined by the connection ∇, uniquely extends to an operator of degree $+1$ on the whole graded algebra $\mathcal{E} \otimes_A \Omega^\bullet A$ by the formula:

$$\nabla(s \otimes \omega) = \nabla s \otimes \omega + s \otimes d\omega$$

where $s \in \mathcal{E}$, $\omega \in \Omega^\bullet A$ and we identify $(\mathcal{E} \otimes_A \Omega^1 A) \otimes_A \Omega^n A$ with $\mathcal{E} \otimes_A \Omega^{n+1} A$.

Considering $\mathcal{E} \otimes_A \Omega^\bullet A$ as a right $(\Omega^\bullet A)$-module we obtain the Leibniz rule of the form

$$\nabla(\sigma\omega) = (\nabla\sigma)\omega + (-1)^k \sigma d\omega$$

where $\sigma \in \mathcal{E} \otimes_A \Omega^k A$, $\omega \in \Omega^\bullet A$.

Proposition 16. *A right A-module admits a connection if and only if it is projective.*

Proof. Consider the exact sequence of right A-modules

$$0 \longrightarrow \mathcal{E} \otimes_A \Omega^1 A \xrightarrow{j} \mathcal{E} \otimes_{\mathbb{C}} A \xrightarrow{m} \mathcal{E} \longrightarrow 0$$

where $j(s \otimes da) = s \otimes a - sa \otimes 1_A$, $m(s \otimes a) = sa$ and $\mathcal{E} \otimes_{\mathbb{C}} A$ is considered as a free A-module with the basis given by a basis of $\mathcal{E}$. With any linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1 A$ we can associate the linear map f which is the right inverse of the map m defined by the formula

$$f(s) := a \otimes 1_A + j(\nabla s).$$

We look for the condition under which this map is a homomorphism of A-modules. We have the following formula

$$f(sa) - f(s)a = j\left(\nabla(sa) - \nabla s \cdot a - s \otimes da\right).$$

Indeed,

$$f(sa) = sa \otimes 1_A + j\left(\nabla(sa)\right),$$
$$f(s)a = (s \otimes 1_A)a + j(\nabla s)a = sa \otimes 1_A + j(\nabla s)a.$$

But $(\nabla s)a = \nabla s \cdot a + s \otimes da$ which implies that

$$f(sa) - f(s)a = j\left(\nabla(sa) - \nabla s \cdot a - s \otimes da\right).$$

So the map f is a homomorphism of A-modules if ∇ satisfies the Leibniz rule. If this rule is fulfilled than the map f splits the above exact sequence which implies that the module $\mathcal{E}$ is a right summand in the free A-module $\mathcal{E} \otimes_{\mathbb{C}} A$ and so is projective. □

Examples of connections

Note, first of all, that on the tensor product of two A-modules $\mathcal{E}$ and $\mathcal{F}$ over a commutative algebra A, provided with connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$ respectively, we can define the connection which is the tensor product of connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$. This connection is defined by the formula

$$\nabla^{\mathcal{E}\otimes_A\mathcal{F}} := \nabla^{\mathcal{E}} \otimes 1_{\mathcal{F}} + 1_{\mathcal{E}} \otimes \nabla^{\mathcal{F}}.$$

Example 11 (connection on A^n)**.** Recall that we denote by A^n the free A-module consisting of columns with entries from A. Then $A^n \otimes_A \Omega^1 A$ is identified with $(\Omega^1 A)^n$ and

$$d\,{}^t(a_1 \dots a_n) := {}^t(da_1 \dots da_n).$$

If ∇ is a connection on A^n then $\nabla - d$ is a A-linear map from A^n to $(\Omega^1 A)^n$, hence ∇ may be written in the form

$$\nabla = d + \alpha$$

where α is an $(n \times n)$-matrix with entries from $\Omega^1 A$. If $\{u_j\}_{j=1}^n$ is the standard basis in A^n then $du_j = 0$ so $\nabla u_j = \sum_{j=1}^n u_j \alpha_{ij}$. The change of basis $\{u_j\}_{j=1}^n \mapsto \{\tilde{u}_j\}_{j=1}^n$, determined by the matrix $b = (b_{ij})$ according to the formula $\tilde{u}_j = \sum_{j=1}^n u_i b_{ij}$, leads to the replacement of the matrix α by the matrix $\tilde{\alpha}$ equal to

$$\tilde{\alpha} = b^{-1}\alpha b + b^{-1} db.$$

Example 12 (Levi-Civita connection)**.** Let $\mathcal{E} = eA^n$ be a finitely generated projective A-module and connection ∇ is given by the composition

$$\mathcal{E} \xrightarrow{i} A^n \xrightarrow{d} A^n \otimes_A \Omega^1 A \xrightarrow{e} \mathcal{E} \otimes_A \Omega^1 A$$

where $i : \mathcal{E} \hookrightarrow A^n$. Identifying $\mathcal{E}$ with a submodule in A^n, we write down the introduced connection in the form

$$\nabla s = e\, ds.$$

The constructed connection is called the *Levi-Civita connection*.

Example 13 (Hermitian connections)**.** Working with C^*-modules it is natural to use the connections ∇ compatible with the inner product $(\cdot\,,\cdot)$, i.e. satisfying the condition

$$(\nabla s, t) + (s, \nabla t) = d(s, t)$$

for all $s, t \in \mathcal{E}$. Here we suppose that the inner product $(\cdot\,,\cdot)$ is extended to $\mathcal{E} \otimes_A \Omega^1 A$ as a sesquilinear pairing with values in $\Omega^1 A$ by the formula:

$$(s, t \otimes adb) := (s, t)adb.$$

In the case of Levi-Civita connection this means that the corresponding idempotent e should be selfadjoint, i.e. a projector.

The difference $\nabla = \nabla_1 - \nabla_2$ of two Hermitian connections on $\mathcal{E}$ belongs to the space of homomorphisms $\mathrm{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \Omega^1 A)$ and is a skew-Hermitian map, i.e.

$$(\nabla s, t) + (s, \nabla t) = 0.$$

If $\mathcal{E} \cong pA^m$, where p is a projector in $\mathrm{Mat}_m(A)$, then

$$pA^m \otimes_A \Omega^1 A \otimes_A {}^m Ap = p\mathrm{Mat}_m(\Omega^1 A)p.$$

The involution in $\Omega^1 A$ is given by the formula $(adb)^* := d(b^*)a^* = d(b^*a^*) - b^*da^*$. So the skew-Hermitian operator $\alpha \in \mathrm{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \Omega^1 A)$ may be identified with a matrix $\alpha \in \mathrm{Mat}_m(\Omega^1 A)$ consisting of 1-forms so that

$$\alpha = p\alpha = \alpha p = p\alpha p$$

where $\alpha^* = -\alpha$. Then the Hermitian connection ∇ will be written in the form $\nabla = pd + \alpha$ where α satisfies the above conditions.

Curvature of a connection

Consider the linear map

$$\nabla^2 : \mathcal{E} \otimes_A \Omega^\bullet A \longrightarrow \mathcal{E} \otimes_A \Omega^{\bullet+2} A.$$

It satisfies the relation

$$\nabla^2(s\omega) = \nabla(\nabla s\, \omega + sd\omega) = (\nabla^2 s)\omega - \nabla s\, d\omega + \nabla s\, d\omega + sd^2\omega = (\nabla^2 s)\omega$$

which means that ∇^2 is a homomorphism of $(\Omega^\bullet A)$-modules which is completely determined by its restriction to $\mathcal{E}$. This homomorphism is called the *curvature* of the connection ∇ and is denoted by K_∇.

Let us compute the curvature of the connection $d+\alpha$ in the free module A^n. We have

$$K_\nabla s = \nabla(ds + \alpha s) = d^2 s + d(\alpha s) + \alpha ds + \alpha^2 s = \\ = d\alpha\, s - \alpha ds + \alpha ds + \alpha^2 s = (d\alpha + \alpha^2)s.$$

Lemma 14. *Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be projective modules over a commutative algebra A provided with connections ∇_0 and ∇_1 with curvatures K_0 and K_1 respectively. Then there is an associated connection ∇ in the A-module $Hom_A(\mathcal{E}_0, \mathcal{E}_1)$ given by the formula*

$$(\nabla T)s := \nabla_1(Ts) - T(\nabla_0 s)$$

with curvature

$$\nabla^2 T = K_1 T - T K_0.$$

Proof. Rewrite the formula for ∇ in the form

$$\nabla_1(Ts) = (\nabla T)s + T(\nabla_0 s).$$

Using this formula, it is easy to show that ∇ satisfies the Leibniz rule. Moreover, the same formula implies that

$$\nabla_1\left((\nabla T)s\right) = (\nabla^2 T)s - \nabla T(\nabla_0 s).$$

Hence

$$(\nabla^2 T)s = \nabla_1\left((\nabla T)s\right) + \nabla T(\nabla_0 s) = \\ = \nabla_1\left(\nabla_1(Ts) - T(\nabla_0 s)\right) + \nabla_1\left(T(\nabla_0 s)\right) - T(\nabla_0 s) = \\ = \nabla_1^2 Ts - T\nabla_0^2 s = (K_1 T - TK_0)s.$$

□

In particular, any connection ∇ in $\mathcal{E}$ in the case of a commutative algebra A generates a connection in $\mathrm{End}_A\mathcal{E}$ given by the formula

$$\nabla T := \nabla \circ T - T \circ \nabla.$$

2.2.4 Chern character

Let M be a smooth compact manifold and $E \longrightarrow M$ is a vector bundle over M. Denote by $\Gamma^\infty(M, E) \equiv \Gamma^\infty(E)$ the module of smooth sections of this bundle. Since this bundle may be embedded as the right summand into the trivial bundle of rank N over M this module may be represented in the form $p\,[C^\infty(M)]^N$ where p is a projector in the free module $[C^\infty(M)]^N$.

We provide M with a Riemannian metric g and denote by R the curvature of this metric. If $s \in \Gamma^\infty(E)$ is a smooth section of the bundle $E \longrightarrow M$, i.e. $ps = s$, then

$$Rs = (\nabla_g)^2 s = (pd)(pd)s = pdpds.$$

Differentiating the relation $p^2 = p$, we obtain: $pdp + dp\,p = dp$, whence

$$pdp = dp(1-p) \quad \text{and} \quad dp\,p = (1-p)dp$$

so that $pdp\,p = 0$. If $s = ps$ then $ds = dp\,s + pds$ which implies that $dp\,s = (1-p)ds$.

From these relations we get

$$Rs = pdpds = dp(1-p)ds = dpdp\,s = dpdp\,ps.$$

Hence,

$$R = dpdp\,p = dp(1-p)dp = pdpdp.$$

Definition 47. The *Chern character* of a projector $p \in \mathrm{Mat}_m(A)$ over a commutative algebra A is the following quantity

$$\mathrm{ch}\,p := \mathrm{tr}(\exp R) = \sum_{k=0}^{\infty} \mathrm{ch}_{2k}(p) := \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{tr}\,p(dp)^{2k}.$$

In the case when A coincides with $C^\infty(M)$ we can consider dp as a matrix consisting of 1-forms, i.e. $dp \in \mathrm{Mat}_N(\Omega^1(M))$, so that every term $\mathrm{ch}_{2k}(p) \in \Omega^{2k}(M)$, in particular, the sum in the definition of the character is finite.

Proposition 17. *The Chern character in the case of algebra $A = C^\infty(M)$ represents a de Rham cohomology class.*

Proof. For the proof of proposition we should check that all forms $\mathrm{ch}_{2k}(p)$ are closed. Indeed,

$$d\left(\mathrm{tr}\,p(dp)^{2k}\right) = \mathrm{tr}\,(dp)^{2k+1} = \mathrm{tr}\,\left(p(dp)^{2k+1}\right) + \mathrm{tr}\,\left((1-p)(dp)^{2k+1}\right).$$

Both terms in the right hand side are equal to zero. For instance,

$$\mathrm{tr}\,\left(p(dp)^{2k+1}\right) = \mathrm{tr}\,\left(p^2(dp)^{2k+1}\right) = \mathrm{tr}\,\left(p(dp)^{2k+1}p\right) = \mathrm{tr}\,\left(p(1-p)(dp)^{2k+1}\right) = 0.$$

□

Proposition 18. *The Chern class $[ch_{2k}(p)]$ depends only on the class $[p]$ in the group $K_0(A)$.*

Proof. It is sufficient to show that the map $p \mapsto \mathrm{ch}_{2k}(p)$ is homotopy invariant. Indeed, let $\{p_t\}$, $0 \le t \le 1$, is a smooth family of projectors. Denote: $\dot{p}_t := \frac{dp_t}{dt}$. We have to show that the form

$$\frac{d}{dt}\mathrm{tr}\left(p_t(dp_t)^{2k}\right) = \mathrm{tr}\left(\dot{p}_t(dp_t)^{2k}\right) + \mathrm{tr}\left(p_t\frac{d}{dt}(dp_t)^{2k}\right)$$

is exact.

Let us show that the first summand in the right hand side vanishes. Indeed, as in the formulas in the beginning of this section, we have: $p_t\dot{p}_t = \dot{p}_t(1-p_t)$ $p_t\dot{p}_tp_t = 0$. Hence the expressions in the right hand side of the following formula

$$\mathrm{tr}\left(\dot{p}_t(dp_t)^{2k}\right) = \mathrm{tr}\left(p_t\dot{p}_t(dp_t)^{2k}\right) + \mathrm{tr}\left((1-p_t)\dot{p}_t(dp_t)^{2k}\right)$$

vanish. For instance,

$$\mathrm{tr}\left(p_t\dot{p}_t(dp_t)^{2k}\right) = \mathrm{tr}\left(p_t\dot{p}_t(dp_t)^{2k}p_t\right) = \mathrm{tr}\left((1-p_t)p_t\dot{p}_t(dp_t)^{2k}\right) = 0.$$

On the other side,

$$\mathrm{tr}\left(p_t\frac{d}{dt}(dp_t)^{2k}\right) = \sum_{j=0}^{2k-1}\mathrm{tr}\left(p_t(dp_t)^j\frac{d}{dt}(dp_t)(dp_t)^{2k-j-1}\right) =$$

$$= \sum_{\cdot j}\mathrm{tr}\left((dp_t)^{2k-1}p_t\frac{d}{dt}(dp_t)\right) + \sum_{\cdot j}\mathrm{tr}\left((dp_t)^{2k-1}(1-p_t)\frac{d}{dt}(dp_t)\right) =$$

$$2k\mathrm{tr}\left((dp_t)^{2k-1}\frac{d}{dt}(dp_t)\right) = \frac{d}{dt}\mathrm{tr}(dp_t)^{2k},$$

i.e. in the right hand side we have an exact form which implies the required assertion. □

Proposition 19. *The Chern character is the ring homomorphism from $K_0(C^\infty(M))$ to $H^{ev}_{dR}(M)$.*

Proof. We have to check that for arbitrary vector bundles E, F over M the following properties

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}\,E + \mathrm{ch}\,F, \quad \mathrm{ch}(E \otimes F) = (\mathrm{ch}\,E))(\mathrm{ch}\,F)$$

hold. The first equality follows from

$$\mathrm{ch}_{2k}(p \oplus q) = \frac{1}{k!}\mathrm{tr}\left(p(dp)^{2k} \oplus q(dq)^{2k}\right) = \mathrm{ch}_{2k}(p) + \mathrm{ch}_{2k}(q).$$

To prove the second equality we use the evident relation

$$\Gamma^\infty(E \otimes F) \cong (p \otimes q)A^{mn}$$

where $A = C^\infty(M)$. Note that the curvature of the tensor product $\nabla^{E\otimes F} = \nabla^E \otimes 1 + 1 \otimes \nabla^F$ of connections ∇^E on E and ∇^F on F is equal to $K_{\nabla^E} \otimes 1 + 1 \otimes K_{\nabla^F}$. Hence

$$\mathrm{ch}(p \otimes q) = \mathrm{tr}\left(\exp(pdpdp \otimes 1 + 1 \otimes qdqdq)\right) = (\mathrm{ch}\,p)(\mathrm{ch}\,q).$$

□

2.2.5 Hochschild homology and cohomology

Chain complexes

Let $(C_\bullet, d)$ be a chain complex of Abelian groups.

Definition 48. A chain map $f : (C_\bullet, d) \to (C'_\bullet, d')$ from a chain complex $(C_\bullet, d)$ to another chain complex $(C'_\bullet, d')$ is a family of maps $f_n : C_n \to C'_n$ making the following diagram commutative:

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ C'_n & \xrightarrow[d'_{n-1}]{} & C'_{n-1}\,. \end{array}$$

The maps f_n send cycles to cycles and boundaries to boundaries, hence they induce homomorphisms $H_n f : H_n(C) \to H_n(C')$.

Definition 49. A *chain homotopy* between two chain maps $f, g : (C_\bullet, d) \to (C'_\bullet, d')$ is a sequence of maps $h_n : C_n \to C'_{n+1}$ satisfying the relations

$$h_{n-1}d_n + d'_{n+1}h_n = f_n - g_n$$

which may be written in the concise form: $hd + d'h = f - g$.

If the maps f, g are chain homotopic then $H_n f = H_n g$ since the closedness of the chain c: $dc = 0$, implies the equality: $f(c) - g(c) = d'h(c)$.

Definition 50. A chain complex $(C_\bullet, d)$ is called *acyclic* if

$$H_n(C) = 0 \quad \text{for } n > 0.$$

Let now $(C_\bullet(A), b)$ be a chain complex of algebras $C_n(A) = A^{\otimes(n+1)}$, where A is a unital algebra, with the boundary map b given on $C_n(A)$ by the formula

$$b_n(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} \quad (2.6)$$

and $b_0 = 0$ on $C_0(A) = A$. For instance, $b_1(a_0 \otimes a_1) = a_0 a_1 - a_1 a_0$ and

$$b_2(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1.$$

It is easy to show that $b_n b_{n-1} = 0$ for any n.

Hochschild homology

Definition 51. The *Hochschild homology* of an algebra A is the homology of the complex $(C_\bullet(A), b)$ denoted by $H_\bullet(A, A)$ or $HH_\bullet(A)$.

Lemma 15. $HH_0(\mathbb{C}) = \mathbb{C}$, $HH_n(\mathbb{C}) = 0$ *for* $n > 0$.

Proof. For the algebra $A = \mathbb{C}$ we have: $C_n(\mathbb{C}) = \mathbb{C}^{\otimes(n+1)} \cong \mathbb{C}$ and $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ coincides with the usual product $a_0 a_1 \ldots a_n$. The boundary map is determined by the formula:

$$b_n(1) = \sum_{j=0}^{n} (-1)^j = \begin{cases} 1 & \text{for even } n \\ 0 & \text{for odd } n. \end{cases}$$

Thus, in this case the Hochschild complex reduces to the exact sequence

$$\ldots \xrightarrow{1} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} \mathbb{C}$$

and so has trivial homology in all dimensions except for $n = 0$. □

In analogous way one can define the Hochschild homology of an algebra A with values in an arbitrary A-bimodule $\mathcal{E}$. For that it is sufficient to put $C_n(A, \mathcal{E}) := \mathcal{E} \otimes_A A^n$. The homology of the obtained complex are denoted by $H_\bullet(A, \mathcal{E})$.

Any homomorphism of algebras $f : A \to A'$ generates the chain map and homomorphism $HH_\bullet f : HH_\bullet(A) \to HH_\bullet(A')$ of zero degree. In other words, HH_n is a functor from the category of unital algebras into the category of vector spaces.

The chains of the form $a = a_0 \otimes \ldots \otimes 1_A \otimes \ldots \otimes a_n$ with $a_k = 1_A$ for some k generate a subcomplex $D_\bullet A$ since $a \in D_n A$ implies that $b_n a \in D_{n-1} A$ (why?). We introduce the boundary maps b'_n given by the formula

$$b'_n(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n$$

which is obtained from the Formula (2.6) for the boundary map b_n by cancelling the last term. Taking the composition of b'_{n+1} with the map $s_0 : a_0 \otimes \ldots \otimes a_n \mapsto 1_A \otimes a_0 \otimes \ldots \otimes a_n$, we get the formula

$$b'_{n+1} s_0(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_n + \sum_{j=0}^{n-1} (-1)^{j+1} 1_A \otimes a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n =$$
$$= (1 - s_0 b'_n)(a_0 \otimes \ldots \otimes a_n).$$

In particular, it implies that $(b_{n+1} s_0 + s_0 b_n)a = a$ for any $a \in D_n A$ with $a_n = 1$. Applying in analogous way the map s_j, inserting the unit 1_A at the jth place in $a_0 \otimes \ldots \otimes a_n$, we obtain that $(b_{n+1} s_j + s_j b_n)a = a$ for any $a \in D_n A$ with $a_j = 1$. Thus, the family of maps $\{s_j\}$ determines a chain homotopy between the zero and identity maps on $D_n(A)$. Hence, the complex $D_\bullet(A)$ is acyclic, i.e. $H_n(D_\bullet(A), b) = 0$ for $n > 0$.

Using that we can define the quotient complex $C_\bullet(A)/D_\bullet(A)$ coinciding in fact with $\Omega^\bullet(A)$. The boundary operator b on $\Omega^\bullet(A)$ takes on the following form

$$b_n(a_0 da_1 \ldots da_n) = a_0 a_1 da_2 \ldots da_n + \sum_{j=0}^{n-1} (-1)^j a_0 da_1 \ldots d(a_j a_{j+1}) \ldots da_n +$$
$$(-1)^n a_n a_0 da_1 \ldots a_{n-1}.$$

Otherwise, it can be written as

$$b_k(\omega^k da) = (-1)^k [\omega^k, a] \tag{2.7}$$

for $\omega^k \in \Omega^k A$, $a \in A$.

Example 14 (homology H_0)**.** $H_0(A, \mathcal{E}) = \mathcal{E}/[\mathcal{E}, A]$ since the boundary map $b : \mathcal{E} \otimes A \to \mathcal{E}$ is given by the formula: $b(a \otimes a) = sa - as$. In particular, $HH_0(A) = A/[A, A]$ which coincides with A if the algebra A is commutative. If $\mathcal{E}$ is a symmetric bimodule over a commutative algebra A then $H_0(A, \mathcal{E}) = \mathcal{E}$.

Hochschild cohomology

Definition 52. A *Hochschild n-cochain* on an algebra A is an $(n+1)$-linear functional on the algebra A or n-linear form on A with values in the dual space A'. Note that A' is an A-bimodule with respect to the operation

$$\varphi \in A' \longmapsto (b\varphi c)(a) := \varphi(cab).$$

The coboundary operator b is dual to the homology boundary operator:

$$b_n\varphi(a_0, \ldots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j \varphi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + \\ + (-1)^{n+1}\varphi(a_{n+1}a_0, \ldots, a_n).$$

The cohomology of the obtained cochain complex are called the *Hochschild cohomology* of the algebra A and denoted by $HH^\bullet(A)$ or $H^\bullet(A, A')$.

In particular, a 0-cocycle τ on the algebra A coincides with the trace since $\tau \in A' = \mathrm{Hom}(A, \mathbb{C})$ and

$$\tau(a_0 a_1) - \tau(a_1 a_0) =: b_1\tau(a_0, a_1) = 0.$$

In a more general way, one can define the Hochschild cohomology of the algebra A with values in an arbitrary A-bimodule $\mathcal{E}$. For that we denote by $C^n(A, \mathcal{E})$ the vector space of n-linear maps $\varphi : A^n \to \mathcal{E}$ considered as an A-bimodule with respect to the operation: $(b\varphi c)(a_1, \ldots, a_n) := b\varphi(a_1, \ldots, a_n)c$ where $b, c \in A$. The coboundary map in this case is given by the formula

$$b_n\varphi(a_1, \ldots, a_{n+1}) = a_1\varphi(a_2, \ldots, a_{n+1}) + \sum_{j=0}^{n} (-1)^j \varphi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + \\ + (-1)^{n+1}\varphi(a_1, \ldots, a_n)a_{n+1}.$$

Definition 53. An n-cochain φ on an algebra A is called *cyclic* if $\lambda\varphi = \varphi$ where

$$\lambda\varphi(a_0, \ldots, a_n) := (-1)^n \varphi(a_n, a_0, \ldots, a_{n-1}).$$

For example, a cyclic 1-cocycle φ satisfies the relations: $\varphi(a_0, a_1) = -\varphi(a_1, a_0)$

$$\varphi(a_0 a_1, a_2) - \varphi(a_0, a_1 a_2) + \varphi(a_2 a_0, a_1) = 0,$$

while a cyclic 1-coboundary $\varphi = b\psi$ is determined by the equality

$$\varphi(a_0, a_1) = b\psi(a_0, a_1) = \psi([a_0, a_1]),$$

i.e. it is a linear function of commutators.

Chern character

Definition 54. Suppose that it is given an n-dimensional cycle $(\Omega^\bullet, d, \int)$ over an algebra A. Its *Chern character* is an $(n+1)$-linear functional on A given by the formula

$$\tau(a_0, \dots, a_n) := \int a_0 da_1 \dots da_n.$$

Note first of all that τ is a cocycle, i.e. $b\tau = 0$. Indeed,

$$\int \sum_{j=0}^{n} (-1)^j a_0 da_1 \dots d(a_j a_{j+1}) \dots da_{n+1}) + (-1)^{n+1} \int a_{n+1} a_0 da_1 \dots da_n =$$

$$= (-1)^n \int (a_0 da_1 \dots da_n) a_{n+1} + (-1)^{n+1} \int (a_{n+1} a_0 da_1 \dots da_n) = 0$$

since $\int a\omega^n = \int \omega^n a$ for any $a \in A$, $\omega^n \in \Omega^n$.

We note next that the cocycle τ is cyclic since

$$\tau(a_0, a_1, \dots, a_n) = (-1)^{n-1} \int da_n a_0 da_1 \dots da_{n-1} =$$

$$= (-1)^n \int a_n da_0 da_1 \dots da_{n-1} = (-1)^n \tau(a_n, a_0, \dots, a_{n-1}).$$

Moreover, $\tau(1, a_1, \dots, a_n) = \int da_1 \dots da_n = 0$.

Proposition 20. *An $(n+1)$-linear functional $\tau : A^{n+1} \to \mathbb{C}$, vanishing on $\mathbb{C} \oplus A^n$, is a cyclic n-cocycle if and only if it coincides with the Chern character of some n-dimensional cycle over A.*

Proof. We have shown already that the Chern character of n-dimensional cycle over A has these properties. Conversely, if an $(n+1)$-linear functional $\tau : A^{n+1} \to \mathbb{C}$ is a cyclic cocycle, vanishing on $\mathbb{C} \oplus A^n$, then we can construct an n-dimensional cycle over A, for which this cocycle will coincide with its Chern character, in the following way.

Let $\Omega^\bullet = \bigoplus_{k=0}^n \Omega^k A$ be the universal DG-algebra with the universal differential d on $\Omega^k A$ for $k < n$. We extend this definition to $k = n$ by setting: $d|\Omega^n A = 0$. We define next the integral $\int : \Omega^n A \to \mathbb{C}$ by setting

$$\int a_0 da_1 \dots da_n := \tau(a_0, a_1, \dots, a_n).$$

We shall prove that this integral indeed determines the Chern character by showing that $(\Omega^\bullet, d, \int)$ is an n-dimensional cycle over A.

We have: $\Omega^n A = A \otimes \bar{A}^{\otimes n}$ which implies that the form $a_0 da_1 \dots da_n$ does not change if some of its elements a_j with $1 \le j \le n$ is replaced by $a_j + \lambda_j 1_A$ with $\lambda_j \in \mathbb{C}$. In order to show that the introduced integral is correctly defined we have to check that $\tau(a_0, a_1, \dots, a_n) = 0$ if one of the elements $a_j = 1$. But this follows from the relation $\tau(1, a_1, \dots, a_n) = 0$ which is satisfied by the hypothesis and cyclicity of τ. The same relation shows that $\int da_1 \dots da_n = 0$ (closedness of $\int$).

It remains to check the permutation property:

$$\int \omega^k \omega^{n-k} = (-1)^{k(n-k)} \int \omega^{n-k} \omega^k.$$

Consider first the case when $k = n$ assuming that $\omega^0 =: a \in A$. By Formula (2.7) we have:

$$\omega^n a - a\omega^n = [\omega^n, a] = (-1)^n b(\omega^n da).$$

Since $b\tau = 0$ it implies that $\int \omega^n a = \int a\omega^n$.

If $\omega^{n-1} \in \Omega^{n-1}A$ and $da \in \Omega^1 A$ then

$$\begin{aligned}\omega^{n-1}da - (-1)^{n-1}da\omega^{n-1} &= [\omega^{n-1}, da] = \\ &= (-1)^{n-1}\left(d[\omega^{n-1}, a] - [d\omega^{n-1}, a]\right) = (-1)^{n-1}d[\omega^{n-1}, a] + b(d\omega^{n-1}da).\end{aligned}$$

Since $b\tau = 0$ and the integral $\int$ is closed we have

$$\int \omega^{n-1}da = (-1)^{n-1}\int da\omega^{n-1}.$$

Using successively these two cases we show also in the general case that

$$\begin{aligned}\int \omega^{n-k}a_0da_1\dots da_k &= (-1)^{n-1}\int da_k\omega^{n-k}da_1\dots da_{k-1} = \dots \\ &(-1)^{k(n-1)}\int a_0da_1\dots da_k\omega^{n-k} = (-1)^{k(n-k)}\int a_0da_1\dots da_k\omega^{n-k}.\end{aligned}$$

□

Chapter 3

SPINOR GEOMETRY

3.1 Spinor algebra

3.1.1 Clifford algebras

Definition

Let V be an n-dimensional Euclidean space provided with an orthonormal basis $\{e_i\}_{i=1}^n$.

Definition 55. A *Clifford algebra* is an associative algebra $\mathrm{Cl}(V)$ over the field $\mathbb{R}$ with generators $1, e_1, \dots, e_n$ satisfying the relations

$$e_i^2 = -1, \quad e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j.$$

We shall also denote it by $\mathrm{Cl}(n)$.

It follows from the given definition that $V \subset \mathrm{Cl}(V)$ and

$$uv + vu = -2(u, v) \quad \text{for } u, v \in V.$$

As a real vector space, $\mathrm{Cl}(V)$ has dimension 2^n and can be provided with the basis given by 1 and elements of the form

$$e_I := e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k}$$

where $I = \{i_1, i_2, \dots, i_k\}$ is a strictly increasing subset of indices with $|I| := k$ elements taken from the set $\{1, 2, , \dots, n\}$, i.e. $1 \leq i_1 < \ldots < i_k \leq n$. In particular, any element $x \in \mathrm{Cl}(V)$ can be written in the form

$$x = \sum_I x_I e_I$$

where we add to the collection $\{I\}$ of sets of indices the subset $I = 0$ and put $e_0 := 1$.

Using this representation, we can introduce a natural *inner product* on $\mathrm{Cl}(V)$ defined by the formula

$$(x, y) := \sum_I x_I y_I$$

(which does not depend on the choice of the orthonormal basis $\{e_i\}_{i=1}^n$).

Denote by $\mathrm{Cl}_k(V)$ the subset $\mathrm{Cl}(V)$ consisting of elements of *degree* k which are linear combinations of basis elements e_I with $|I| = k$ (assuming that $I = 0$ for $k = 0$). We introduce also the following subsets of $\mathrm{Cl}(V)$:

$$\mathrm{Cl}^{\mathrm{ev}}(V) := \bigoplus_{k \text{ even}} \mathrm{Cl}_k(V), \quad \mathrm{Cl}^{\mathrm{od}}(V) := \bigoplus_{k \text{ odd}} \mathrm{Cl}_k(V).$$

Then $\mathrm{Cl}^{\mathrm{ev}}(V)$ will be a unital subalgebra in $\mathrm{Cl}(V)$ and

$$\mathrm{Cl}(V) = \mathrm{Cl}^{\mathrm{ev}}(V) \oplus \mathrm{Cl}^{\mathrm{od}}(V)$$

which provides $\mathrm{Cl}(V)$ with the structure of a *superalgebra*.

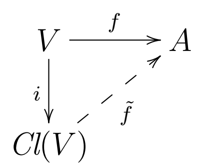
Universal property

The notion of the Clifford algebra $\mathrm{Cl}(V)$ does not depend in fact on the choice of the orthonormal basis $\{e_i\}_{i=1}^n$ due to the following universal property of this algebra which may be taken for its definition.

Proposition 21. *The Clifford algebra $Cl(V)$ is a unique associative $\mathbb{R}$-algebra with unit which contains the Euclidean space V and has the following property: for any associative $\mathbb{R}$-algebra A with unit 1_A and any linear map $f : V \to A$, satisfying the condition*

$$f(v) \cdot f(v) = -|v|^2 \, 1_A,$$

there exists a unique extension of f to an algebra homomorphism $\tilde{f} : Cl(V) \to A$ such that the following diagram



is commutative.

Proof. To prove the formulated universal property of the Clifford algebra we use the following equivalent definition of this algebra. Denote by

$$\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

the *tensor algebra* of the space V and consider the ideal $\mathcal{J}(V)$ of this algebra generated by the elements of the form

$$v \otimes v + |v|^2 \cdot 1.$$

The Clifford algebra $\mathrm{Cl}(V)$ coincides with the quotient of the tensor algebra $\mathcal{T}(V)$ by this ideal:

$$\mathrm{Cl}(V) \cong \mathcal{T}(V)/\mathcal{J}(V)$$

(check this assertion!).

Return to the proof of the universal property of the algebra $\mathrm{Cl}(V)$. Any linear map $f : V \to A$ extends uniquely to an algebra homomorphism

$$\tilde{f} : \mathcal{T}(V) \longrightarrow A.$$

By assumption, this homomorphism vanishes on the ideal $\mathcal{J}(V)$ so $\tilde{f}$ can be pushed down to an algebra homomorphism $\tilde{f} : \mathrm{Cl}(V) \to A$. □

Examples of Clifford algebras

1. $\mathrm{Cl}(\mathbb{R}) = \mathbb{C}$ with $e_1 = i$.
2. $\mathrm{Cl}(\mathbb{R}^2) = \mathbb{H}$ with $e_1 = i, e_2 = j, e_1e_2 = k$.
3. $\mathrm{Cl}(\mathbb{R}^4) = \mathrm{Mat}_2(\mathbb{H})$ is the space of quaternion 2×2-matrices.

Multiplicative group

Denote by $\mathrm{Cl}^\times(V)$ the *group of invertible elements* of the Clifford algebra $\mathrm{Cl}(V)$. It is a Lie group which contains $V \setminus \{0\}$ since for any element $v \in V \setminus \{0\}$ the inverse element v^{-1} can be given by the formula

$$v^{-1} = -\frac{v}{|v|^2}\,.$$

The Lie algebra of the group $\mathrm{Cl}^\times(V)$ is the algebra $\mathrm{cl}(V)$ which coincides as a set with $\mathrm{Cl}(V)$ and is provided with the Lie bracket of the form

$$[x, y] := xy - yx.$$

The group $\mathrm{Cl}^\times(V)$ acts on the algebra $\mathrm{Cl}(V)$ by the *adjoint representation*

$$w \longmapsto \mathrm{Ad}_w x := wxw^{-1}, \quad w \in \mathrm{Cl}^\times(V).$$

The differential of this action is a Lie algebra homomorphism

$$\mathrm{ad} : \mathrm{cl}(V) \longrightarrow \mathrm{Der}\,\mathrm{Cl}(V)$$

from the algebra $\mathrm{cl}(V)$ to the algebra $\mathrm{Der}\,\mathrm{Cl}(V)$ of derivations of $\mathrm{Cl}(V)$ given by the formula

$$\mathrm{ad}_y x := [y, x], \qquad y \in \mathrm{cl}(V),\, x \in \mathrm{Cl}(V).$$

For any $u \in V \setminus \{0\}$ the map

$$-\mathrm{Ad}_u v = v - 2\frac{(u, v)}{|u|^2}u\,, \quad v \in V,$$

is the reflection of V with respect to the hyperplane $u^\perp$ orthogonal to u. In order to get rid of the minus sign in the left hand side of the last equality it is convenient to use, instead of the adjoint representation Ad, the action of the group $\mathrm{Cl}^\times(V)$ on the algebra $\mathrm{Cl}(V)$ given by the *twisted adjoint representation* of the form

$$w \longmapsto \pi_w(x) := \chi(w)xw^{-1}, \quad x \in \mathrm{Cl}(V),\, w \in \mathrm{Cl}^\times(V)$$

where χ is the *grading map* defined on the homogeneous elements of degree k from the group $\mathrm{Cl}^\times(V)$ by the formula

$$\chi(w) := (-1)^{\deg w} w = (-1)^k w.$$

Then the map $\pi_u : V \to V$, determined by an element $u \in V \setminus \{0\}$, will coincide with the *reflection* with respect to the hyperplane $u^\perp$.

Taking into account these remarks, we can consider the subgroup of multiplicative group $\mathrm{Cl}^\times(V)$ consisting of the elements $x \in \mathrm{Cl}^\times(V)$ which have the property: $\pi_x(V) = V$. As it was pointed out before, this property has any element $v \in V \setminus \{0\}$ so it is natural to introduce the following group.

Definition 56. The *Clifford group* $\Gamma(V) \equiv \Gamma(n)$ is the subgroup of multiplicative group $\mathrm{Cl}^\times(V)$ generated by the elements $v \in V \setminus \{0\}$.

Every element of the group $\Gamma(V)$ generates a non-degenerate linear transform of the space V so we have a homomorphism

$$\pi : \Gamma(V) \longrightarrow \mathrm{GL}(V).$$

This homomorphism takes values in the orthogonal group $\mathrm{O}(V)$. Indeed, since any element $x \in \Gamma(V)$ may be represented as the product $x = v_1 \cdot \ldots v_k$, where $v_i \in V \setminus \{0\}$, the corresponding transform π_x is the composition of reflections associated with elements v_i, i.e. belongs to $\mathrm{O}(V)$. Moreover, the homomorphism $\pi : \Gamma(V) \to \mathrm{O}(V)$ is an epimorphism since any orthogonal transform is the composition of reflections.

The homomorphism $\pi : \Gamma(V) \to \mathrm{O}(V)$ may be included into the exact sequence of group homomorphisms of the form

$$1 \longrightarrow \mathbb{R}^\times \longrightarrow \Gamma(V) \xrightarrow{\pi} \mathrm{O}(V) \longrightarrow 1.$$

We set also

$$S\Gamma(V) := \Gamma(V) \cap \mathrm{Cl}^{\mathrm{ev}}(V).$$

Generalization of the Clifford construction to other fields and quadratic forms

The given definition of Clifford algebras immediately extends to the case of vector spaces V over an arbitrary field K (we consider here only the fields $K = \mathbb{R}, \mathbb{C}$) provided with a non-degenerate symmetric bilinear form B.

In this case the Clifford algebra $\mathrm{Cl}(V, B)$ is defined again as

$$\mathrm{Cl}(V, B) = \mathcal{T}(V)/\mathcal{J}(V, B)$$

where the ideal $\mathcal{J}(V, B)$ is generated by the elements of the form

$$u \otimes v + v \otimes u + 2B(u, v) \cdot 1 \quad \text{where } u, v \in V.$$

Otherwise, we can define $\mathrm{Cl}(V, B)$ as an associative algebra with unit generated by the elements $u \in V$ satisfying the relations

$$uv + vu + 2B(u, v) \cdot 1 = 0.$$

This algebra has again the universal property: if A is another associative algebra with unit 1_A and $f : V \to A$ is an arbitrary linear map with the property:

$$f(u)f(v) + f(v)f(u) = -2B(u,v) \cdot 1_A$$

for all $u, v \in V$, then this map extends uniquely to an algebra homomorphism $\tilde{f} : \mathrm{Cl}(V,B) \to A$.

Complexified Clifford algebra

The construction, presented in the previous section, is interesting for us, first of all, in the case when $K = \mathbb{C}$ and the complex vector space is provided with a non-degenerate symmetric bilinear form (defined uniquely up to multiplication by a nonzero complex number).

We introduce also the *complexified Clifford algebra* $\mathbb{C}\mathrm{l}(V)$ of an n-dimensional real vector space V by setting

$$\mathbb{C}\mathrm{l}(V) := \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}.$$

This algebra is isomorphic to the Clifford algebra $\mathrm{Cl}(V^{\mathbb{C}})$ of the complexified vector space $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ provided with the complexified quadratic form.

3.1.2 Spinor groups

The group Pin

Definition 57. The *group* $\mathrm{Pin}(V)$ is defined as the subgroup of the Clifford group $\Gamma(V)$ generated by the unit vectors from V, i.e. by vectors $v \in V$ with $|v| = 1$.

As in the case of the Clifford group, we have a homomorphism

$$\pi : \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V)$$

which is included into the exact sequence of group homomorphisms

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \xrightarrow{\pi} \mathrm{O}(V) \longrightarrow 1.$$

The group Spin

Definition 58. The *group* $\mathrm{Spin}(V)$ is the identity connected component of the group $\mathrm{Pin}(V)$. It can be also defined as

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^{\mathrm{ev}}(V).$$

As in the case of the group $\mathrm{Pin}(V)$, there is an exact sequence of group homomorphisms

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \xrightarrow{\pi} \mathrm{SO}(V) \longrightarrow 1.$$

For $n > 2$ the group $\mathrm{Spin}(n)$ is a simply connected covering group of the group $\mathrm{SO}(V)$.

Examples of Spin-groups

1. $\mathrm{Spin}(2) = \mathrm{U}(1) = \mathrm{SO}(2)$.
2. $\mathrm{Spin}(3) = \mathrm{SU}(2)$.
3. $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$.

3.1.3 Relation to the exterior algebra

Definition

The *exterior algebra*

$$\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$$

of the n-dimensional real vector space V may be also defined as the quotient of the tensor algebra $\mathcal{T}(V)$ by the ideal $I(V)$ generated by the elements of the form

$$u \otimes v + v \otimes u.$$

Otherwise, $\Lambda(V)$ is an associative unital algebra, containing V, with the product $\wedge$ satisfying the relation

$$u \wedge v + v \wedge u = 0 \quad \text{for all } u, v \in V.$$

For a fixed orthonormal basis $\{e_i\}_{i=1}^n$ of V we can define the subspace $\Lambda^k(V)$ of elements of order k generated by 1 and elements of the form

$$\varepsilon_I := e_{i_1} \wedge \ldots \wedge e_{i_k}$$

where $I = \{i_1, \ldots, i_k\}$ is a strictly increasing subset of indices from the set $\{1, \ldots, n\}$: $1 \le i_1 < \ldots < i_k \le n$. We also extend this definition by setting $I = 0$ and $e_0 = 1$ for $k = 0$. The dimension of $\Lambda^k(V)$ is equal to $\binom{n}{k}$ which implies that the dimension of $\Lambda(V)$ is equal to 2^n, i.e. coincides with the dimension of the Clifford algebra $\mathrm{Cl}(n)$. So we can expect that for the n-dimensional Euclidean space V it should exist a close relation between $\Lambda(V)$ and $\mathrm{Cl}(V)$ be established in the next section.

The exterior algebra can be also defined as a graded superalgebra having the following universal property. If A is another graded superalgebra and $f : V \to A^1$ is an arbitrary linear map with the property:

$$f(u)f(v) + f(v)f(u) = 0 \quad \text{for all } u, v \in V$$

then f extends uniquely to a homomorphism of graded superalgebras $\tilde{f} : \Lambda(V) \to A$.

Derivations and transposition

We consider now the derivations of $\Lambda(V)$ paying special attention to the inner and exterior ones.

Define first the *inner product* of a form and an element $\xi \in V'$ of the dual space V' by setting $\iota(\xi)1 = 0$ and

$$\iota(\xi)(v_1 \wedge \ldots \wedge v_k) = \sum_{j=1}^{k} \xi(v_j) v_1 \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_k$$

where $v_1, \ldots, v_k \in V$, and the "hat" over the symbol v_j means that this symbol should be omitted. Then for any $\xi, \eta \in V'$ we shall have

$$\iota(\xi)\iota(\eta) + \iota(\eta)\iota(\xi) = 0.$$

Using now the universal property of the algebra $\Lambda(V')$, we can extend the constructed map $\iota : V' \to \operatorname{End}\Lambda(V)$ to the whole exterior algebra $\Lambda(V')$ obtaining a map $\iota : \Lambda(V') \to \operatorname{End}\Lambda(V)$.

On the other hand, we have on $\Lambda(V)$ the operation of *exterior product* generating a homomorphism $\epsilon : \Lambda(V) \to \operatorname{End}\Lambda(V)$.

These operations are related by the commutation relations:

$$\begin{aligned}
[\epsilon(u), \epsilon(v)] &= \epsilon(u)\epsilon(v) + \epsilon(v)\epsilon(u) = 0, \\
[\iota(\xi), \iota(\eta)] &= \iota(\xi)\iota(\eta) + \iota(\eta)\iota(\xi) = 0, \\
[\iota(\xi), \epsilon(v)] &= \iota(\xi)\epsilon(v) + \epsilon(v)\iota(\xi) = \xi(v)
\end{aligned}$$

fulfilled for all $u, v \in V$, $\xi, \eta \in V'$.

The tensor algebra $\mathcal{T}(V)$ has a special anti-automorphism generating the identity transform on V. This automorphism is called the *transposition* and is uniquely determined by the formula

$$(v_1 \otimes \ldots \otimes v_k)^T := v_k \otimes \ldots \otimes v_1.$$

Since the introduced operation preserves the ideal $I(V)$ it can be pushed down to the exterior algebra $\Lambda(V)$. Moreover,

$$\eta^T = (-1)^{k(k-1)/2}\eta$$

on the forms $\eta \in \Lambda^k(V)$.

In the complex case, if we denote by $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of the space V then we shall have on $V^{\mathbb{C}}$ the operation of complex conjugation: $v \mapsto \bar{v}$ which extends in a natural way to the whole exterior algebra $\Lambda(V^{\mathbb{C}})$. Define the *involution* in $\Lambda(V^{\mathbb{C}})$ by setting

$$\eta \longmapsto \eta^* := (\bar{\eta})^T.$$

This map generates a conjugate-linear anti-automorphism of the algebra $\Lambda(V^{\mathbb{C}})$.

Relation between the exterior algebra and Clifford algebra

Consider the map $s : V \to \operatorname{End}\Lambda(V)$ given by the formula

$$s(v) = \epsilon(v) + \iota(v'), \quad v \in V,$$

where $v' \in V'$ is the functional dual to v which is defined as

$$v' : u \in V \longmapsto (v, u).$$

The map s extends to a homomorphism of the whole tensor algebra $s : \mathcal{T}(V) \to \operatorname{End}\Lambda(V)$. The commutation relations for the inner and exterior derivations imply that

$$s(u)s(v) + s(v)s(u) = -2(u, v)1.$$

Hence s vanishes on the ideal $\mathcal{J}(V)$ and so can be pushed down to a homomorphism

$$s : \mathrm{Cl}(V) \longrightarrow \operatorname{End}\Lambda(V).$$

Consider next the map $\sigma : \mathrm{Cl}(V) \to \Lambda(V)$ called the *symbol* and given by the formula

$$\sigma(x) := s(x)1$$

where 1 is considered as an element of $\Lambda^0(V)$. In low degrees this map is easy to compute explicitly:

$$\begin{aligned}
&\sigma(1) = 1,\\
&\sigma(v) = v,\\
&\sigma(v_1 v_2) = v_1 \wedge v_2 + (v_1, v_2),\\
&\sigma(v_1 v_2 v_3) = v_1 \wedge v_2 \wedge v_3 + (v_2, v_3)v_1 - (v_1, v_3)v_2 + (v_1, v_2)v_3.
\end{aligned}$$

The inverse map of σ, sending

$$\mathrm{Alt} : \Lambda(V) \longrightarrow \mathrm{Cl}(V),$$

is called the *alternation* and is given by the formula

$$\mathrm{Alt}(v_1 \wedge \ldots \wedge v_k) = \frac{1}{k!} \sum_{p \in S_k} (-1)^{\operatorname{sgn} p} v_{p(1)} \cdot \ldots \cdot v_{p(k)}$$

where the summation is taken over all permutations $p \in S_k$ of the set $\{1, \ldots, k\}$, and $\operatorname{sgn} p$ denotes the parity of permutation p.

Involution and volume element

The transposition map on the tensor algebra $\mathcal{T}(V)$, introduced in Sec.3.1.3, preserves the ideal $\mathcal{J}(V)$ and so can be pushed down to an anti-automorphism of the Clifford algebra $\mathrm{Cl}(V)$ having the property:

$$(v_1 \cdot \ldots \cdot v_k)^T = v_k \cdot \ldots \cdot v_1.$$

Note that the alternation map Alt intertwines the transpositions in $\Lambda(V)$ and $\mathrm{Cl}(V)$.

In the complex case we can define the *involution* on the Clifford algebra $\mathbb{C}\mathrm{l}(V)$ by the formula:

$$x \longmapsto x^* := (\bar{x})^T$$

where $x \mapsto \bar{x}$ is the complex conjugation on $\mathbb{C}\mathrm{l}(V)$. Again the alternation map Alt intertwines the involutions in $\Lambda(V)$ and $\mathbb{C}\mathrm{l}(V)$.

Using the transposition, we can introduce one more useful map

$$\mathrm{N} : x \longmapsto x^T x$$

defined on elements x of the Clifford group $\Gamma(V)$. Since any element $x \in \Gamma(V)$ may be represented in the form $x = v_1 \cdot \ldots \cdot v_k$ it follows that N takes values in $\mathbb{R}^\times$. This map generates a group homomorphism

$$\mathrm{N} : \Gamma(V) \longrightarrow \mathbb{R}^\times, \quad x \longmapsto x^T x$$

called the *norm*, having the property: $\mathrm{N}(\lambda x) = \lambda^2 \mathrm{N}(x)$ for $\lambda \in \mathbb{R}^\times$.

The alternation of the volume element $\omega \in \Lambda^n(V)$ of the n-dimensional vector space V yields an element $\omega \in \mathrm{Cl}(V)$ (denoted by the same letter ω) which is called the *volume element* of the Clifford algebra or its *chiral element*.

The square of ω is a scalar which is easy to show by choosing an orthonormal basis $\{e_i\}_{i=1}^n$ of the Euclidean space V and setting $\omega = e_1 \cdot \ldots \cdot e_n$. Then

$$\omega^2 = (-1)^{n(n+1)/2}.$$

Moreover, for odd n the element ω is central while for even n we have: $x\omega = \omega\chi(x)$ for any $x \in \mathrm{Cl}(n)$.

3.1.4 The group Spin^c

Definition

Consider the complexified Clifford algebra $\mathbb{C}\mathrm{l}(V) = \mathrm{Cl}(V^{\mathbb{C}})$, where $V^{\mathbb{C}}$ is the complexification of the space V, and provide it with the *Hermitian inner product* by setting it equal to

$$\langle z, w \rangle := (\bar{z}, w)$$

on elements $z, w \in V^{\mathbb{C}}$ and extending then to the whole algebra $\mathrm{Cl}(V^{\mathbb{C}})$. Recall that this algebra is provided with the involution defined by the equality: $z^* = (\bar{z})^T$ on elements $z \in \mathbb{C}\mathrm{l}(V)$.

Let $\Gamma(V^{\mathbb{C}})$ be the Clifford group of the space $V^{\mathbb{C}}$ consisting of elements $z \in \mathrm{Cl}^\times(V^{\mathbb{C}})$ satisfying the condition: $\pi_z(V^{\mathbb{C}}) = V^{\mathbb{C}}$ so that $\pi_z \in \mathrm{O}(V^{\mathbb{C}})$.

Definition 59. Introduce the following groups:

$$\Gamma^c(V) = \{z \in \Gamma(V^{\mathbb{C}}) : \ z^* z \in \mathbb{R}_+\},$$
$$\mathrm{Pin}^c(V) = \{z \in \Gamma(V^{\mathbb{C}}) : \ z^* z = 1\},$$
$$\mathrm{Spin}^c(V) = \mathrm{Pin}^c(V) \cap \mathrm{S}\Gamma(V^{\mathbb{C}}).$$

It can be shown that the element $z \in \Gamma(V^{\mathbb{C}})$ belongs to the subgroup $\Gamma^c(V)$ if and only if the transform $\pi_z \in \mathrm{O}(V^{\mathbb{C}})$ preserves the real subspace V. In other words, $\Gamma^c(V)$ coincides with the inverse image (with respect to π) of the subgroup $\mathrm{O}(V) \subset \mathrm{O}(V^{\mathbb{C}})$.

The exact sequence for the group $\Gamma(V^{\mathbb{C}})$ transforms into the exact sequence of group homomorphisms

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \Gamma^c(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1$$

inducing the exact sequences

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Pin}^c(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1,$$
$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1,$$

since $\mathbb{C}^\times \cap \mathrm{Pin}^c(V) = \mathbb{C}^\times \cap \mathrm{Spin}^c(V) = \mathrm{U}(1)$. It follows from these sequences that the groups $\mathrm{Pin}^c(V)$ and $\mathrm{Spin}^c(V)$ are central extensions of the groups $\mathrm{O}(V)$ and $\mathrm{SO}(V)$ respectively by $\mathrm{U}(1)$.

Exact sequences for the group $\mathrm{Spin}^c(V)$

One can define the group $\mathrm{Spin}^c(V)$ in a different way as the subgroup of $\Gamma^c(V)$ of the form

$$\mathrm{Spin}^c(V) = \{z = xe^{i\theta} : \ x \in \mathrm{Spin}(V),\, \theta \in \mathbb{R}\}.$$

In other words, the group $\mathrm{Spin}^c(V)$ is the quotient of the group $\mathrm{Spin}(V) \times \mathrm{U}(1)$ by the equivalence relation: $(x, e^{i\theta}) \sim (-x, -e^{i\theta})$. The latter assertion may be written in the form of the exact sequence

$$1 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{Spin}^c(V) \stackrel{\delta}{\longrightarrow} \mathrm{U}(1) \longrightarrow 1$$

where $\delta : xe^{i\theta} \mapsto e^{2i\theta}$. Otherwise,

$$\mathrm{Spin}^c(V) = \mathrm{Spin}(V) \times_{\mathbb{Z}_2} \mathrm{U}(1).$$

The norm homomorphism

$$\mathrm{N} : \Gamma^c(V) \longrightarrow \mathbb{C}^\times, \quad z \longmapsto z^T z,$$

defined on the Clifford group $\Gamma(V^{\mathbb{C}})$, takes values in $\mathrm{U}(1)$ when applied to the elements from $\mathrm{Pin}^c(V)$. The combination of this homomorphism with the above exact sequences yields:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \Gamma^c(V) \stackrel{(\pi,\mathrm{N})}{\longrightarrow} \mathrm{O}(V) \times \mathbb{C}^\times \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}^c(V) \stackrel{(\pi,\mathrm{N})}{\longrightarrow} \mathrm{O}(V) \times \mathrm{U}(1) \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(V) \stackrel{(\pi,\mathrm{N})}{\longrightarrow} \mathrm{SO}(V) \times \mathrm{U}(1) \longrightarrow 1$$

where the homomorphism π is given by the map $z \mapsto \pi_z(v) = \chi(z)vz^{-1}$.

The Lie algebra of the group $\mathrm{Spin}^c(V)$ coincides with

$$\mathrm{spin}^c(V) = \mathrm{cl}_2(V) \oplus i\mathbb{R}$$

where $\mathrm{cl}_2(V)$ is the quadratic component (coinciding as a set with $\mathrm{Cl}_2(V)$) of the Clifford algebra $\mathrm{Cl}(V)$ provided with the commutator as the Lie bracket.

Examples of the Spin^c-groups

1. $\mathrm{Cl}(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C}$, the group $\mathrm{Spin}^c(\mathbb{R})$ coincides with the group $\mathrm{U}(1)$ embedded into $\mathbb{C} \oplus \mathbb{C}$ with the help of diagonal map.
2. $\mathrm{Cl}(\mathbb{R}^2) = \mathrm{Mat}_2(\mathbb{C})$, the group $\mathrm{Spin}^c(\mathbb{R}^2)$ coincides with the group $\mathrm{U}(1) \times \mathrm{U}(1)$ consisting of the unitary diagonal matrices in $\mathrm{Mat}_2(\mathbb{C})$.

3.1.5 Spin representation

Clifford modules

Definition 60. A *Clifford representation* is a homomorphism

$$\rho : \mathrm{Cl}(V) \longrightarrow \mathrm{End}\, S$$

from the Clifford algebra $\mathrm{Cl}(V)$ into the algebra of linear operators acting in a complex vector space S called the *Clifford module* over $\mathrm{Cl}(V)$ or the *spinor space* for the algebra $\mathrm{Cl}(V)$. We shall assume that S is provided with an Hermitian inner product.

Otherwise, the Clifford representation may be defined as a linear map $\rho : V \to \mathrm{End}\, S$ having the property:

$$\rho(u)\rho(v) + \rho(v)\rho(u) + 2(u,v)1 = 0$$

for all $u, v \in V$. Then by universal property such map extends to a representation $\rho : \mathrm{Cl}(V) \to \mathrm{End}\, S$.

The standard definitions and properties from the representation theory of associative algebras apply also to Clifford representations.

The action of the representation ρ on the space S is often denoted by

$$\rho(x)s := x \cdot s$$

for $x \in \mathrm{Cl}(V)$, $s \in S$, and called the *Clifford multiplication*.

Semispinor spaces

The Clifford algebra $\mathrm{Cl}(n)$ of n-dimensional Euclidean vector space V, provided with an orthonormal basis $\{e_i\}_{i=1}^n$, has the volume element $\omega = e_1 \cdot \ldots \cdot e_n$. In the complex case we can introduce, together with ω, also a *complex volume element* ω^c defined by the formula

$$\omega^c := i^{[(n+1)/2]}\omega$$

where $[\,\cdot\,]$ denotes the integral part of a number.

For odd n the elements ω and ω^c belong to the center of the Clifford algebra. Moreover,

$$\begin{aligned} \omega^2 &= 1 \quad \text{for } n \equiv 3, 4 \bmod 4, \\ (\omega^c)^2 &= 1 \quad \text{for all } n. \end{aligned}$$

Consider first the real volume element and suppose that $n \equiv 3, 4 \bmod 4$ so that $\omega^2 = 1$. The the elements $\pi_\pm := \frac{1 \pm \omega}{2}$ are mutually orthogonal idempotents and $\pi_+ + \pi_- = 1$. Hence the Clifford algebra $\mathrm{Cl}(V)$ admits the decomposition of the form

$$\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V)$$

where $\mathrm{Cl}^\pm(V) := \pi_\pm \mathrm{Cl}(V)$.

In the same way any Clifford module S over $\mathrm{Cl}(V)$ can be represented in the form

$$S = S^+ \oplus S^-$$

where $S^{\pm}$ are the eigenspaces of the operator $\rho(\omega)$ with eigenvalues ± 1 so that $S^{\pm} = \pi_{\pm} S$. The subspaces $S^{\pm}$ are called the *semispinor spaces*.

In the complex case for any odd n we shall have an analogous decomposition of the complexified Clifford algebra

$$\mathbb{C}\mathrm{l}(V) = \mathbb{C}\mathrm{l}^{+}(V) \oplus \mathbb{C}\mathrm{l}^{-}(V)$$

where $\mathbb{C}\mathrm{l}^{\pm}(V) := (1 \pm \omega^{c})\mathbb{C}\mathrm{l}(V)$.

Description of irreducible Clifford modules

Proposition 22. *Let $\rho : Cl(n) \to End S$ be an irreducible representation of the Clifford algebra and $n = 4m + 3$. Then*

$$\rho(\omega) = \pm Id.$$

Moreover, both possibilities are realized and the corresponding Clifford representations are not equivalent. An analogous assertion holds for the complexified Clifford algebra $\mathbb{C}l(n)$ for any odd n.

Proof. We represent the spinor space S as the direct sum

$$S = S^{+} \oplus S^{-}$$

of (± 1)-eigenspaces of operator $\rho(\omega)$. These subspaces are invariant under Clifford multiplication because the element $\rho(\omega)$ is central. Since the representation S is irreducible we have either $S = S^{+}$, or $S = S^{-}$ which proves the first assertion of the proposition.

These irreducible representations are not equivalent to each other because if $\rho(\omega) = \pm \mathrm{Id}$, i.e. it is a multiple of Id, then this property is valid also for any equivalent representation ρ_1, moreover $\rho_1(\omega)$ is proportional to Id with the same proportionality coefficient as for $\rho(\omega)$.

In order to see that both possibilities are realized it is sufficient to consider the action of the Clifford algebra $\mathrm{Cl}(V)$ on $\mathrm{Cl}^{\pm}(V)$ by left multiplication.

The proof of proposition in the complex case we leave as an exercise. □

Before passing to the study of irreducible representations of the Clifford algebra in the case $n = 4m$, we prove the following lemma relating the Clifford algebra $\mathrm{Cl}(n-1)$ with the even part of the Clifford algebra $\mathrm{Cl}(n)$.

Lemma 16. *For any $n > 1$ we have an algebra isomorphism*

$$Cl(n-1) \longrightarrow Cl^{ev}(n).$$

Proof. Choose an orthonormal basis $e_1, \ldots, e_{n-1}, e_n$ in the space $\mathbb{R}^n$ and denote by $\mathbb{R}^{n-1}$ the subspace spanned by the first $n-1$ vectors of this basis.

Consider the map $f : \mathbb{R}^{n-1} \to \mathrm{Cl}^{\mathrm{ev}}(n)$ defined on the basis elements by the formula:

$$f(e_i) = e_n \cdot e_i, \; i = 1, \ldots, n-1.$$

Compute the value $f(x)^2$ on an arbitrary element $x = \sum_{i=1}^{n-1} x_i e_i$ from the space $\mathbb{R}^{n-1}$. We have

$$f(x)^2 = \left(\sum_i x_i e_n e_i\right) \cdot \left(\sum_j x_j e_n e_j\right) = \sum_{i,j} x_i x_j e_n e_i e_n e_j =$$

$$\sum_{i,j} x_i x_j e_i e_j \text{ (since } e_n e_i = -e_i e_n \text{ and } e_n^2 = -1\text{)}$$

$$= x \cdot x = -\|x\|^2 \cdot 1.$$

So by the universal property the constructed map extends to an algebra homomorphism

$$f : \mathrm{Cl}(n-1) \longrightarrow \mathrm{Cl}^{\mathrm{ev}}(n).$$

Considering it on the basis elements it is easy to see that this map is an isomorphism. □

Proposition 23. *Let $\rho : Cl(n) \to End\, S$ be an irreducible representation of the Clifford algebra and $n = 4m$. Consider the decomposition of the space S into the direct sum of semispinor spaces*

$$S = S^+ \oplus S^-.$$

Then each of the subspaces $S^\pm$ is invariant under the even part $Cl^{ev}(n)$ of the Clifford algebra $Cl(n)$. These subspaces correspond to two different representations of the algebra $Cl(n-1) \cong Cl^{ev}(n)$.

An analogous assertion holds for the complexified Clifford algebra $\mathbb{C}l(n)$ for any even n.

Proof. The subspaces $S^\pm$ are invariant under $\mathrm{Cl}^{\mathrm{ev}}(n)$ since for even n the volume element ω commutes with any element from $\mathrm{Cl}^{\mathrm{ev}}(n)$. Under the isomorphism $\mathrm{Cl}(n-1) \cong \mathrm{Cl}^{\mathrm{ev}}(n)$ from the proved Lemma 16 the volume element ω' of the algebra $\mathrm{Cl}(n-1)$ is mapped to the volume element $\omega \in \mathrm{Cl}^{\mathrm{ev}}(n)$ of the algebra $\mathrm{Cl}(n)$. Hence, $\rho(\omega') = \mathrm{Id}$ on S^+ and $\rho(\omega') = -\mathrm{Id}$ on S^-. By the previous proposition these representations are not equivalent to each other.

The proof of proposition in the complex case we leave as an exercise. □

Spin representation

Definition 61. The *real spin representation* of the group $\mathrm{Spin}(n)$ is a group homomorphism

$$\Delta_n : \mathrm{Spin}(n) \longrightarrow \mathrm{GL}(S)$$

obtained by the restriction of an irreducible representation $\rho : \mathrm{Cl}(n) \to \mathrm{End}\, S$ of the Clifford algebra $\mathrm{Cl}(n)$ to the group $\mathrm{Spin}(n) \subset \mathrm{Cl}^{\mathrm{ev}}(n) \subset \mathrm{Cl}(n)$.

Proposition 24. *For $n = 4m+3$ the representation Δ_n is irreducible and both irreducible representations of the algebra $Cl(n)$ give the same representation when restricted to $Spin(n)$.*

For $n = 4m$ the representation Δ_{4m} admits the decomposition

$$\Delta_{4m} = \Delta_{4m}^+ \oplus \Delta_{4m}^-$$

into the direct sum of two non-equivalent irreducible representations of the group Spin(n)*.*

Proof. If $n = 4m + 3$ then the grading automorphism χ permutes the summands $\mathrm{Cl}^+(n)$ and $\mathrm{Cl}^-(n)$ in the decomposition

$$\mathrm{Cl}(n) = \mathrm{Cl}^+(n) \oplus \mathrm{Cl}^-(n)$$

since $\chi(\omega) = -\omega$. Hence, the elements of the subalgebra $\mathrm{Cl}^{\mathrm{ev}}(n)$, which are by definition invariant under χ, should have the form $(x, \chi(x)) \in \mathrm{Cl}^+(n) \oplus \mathrm{Cl}^-(n)$, i.e. the subalgebra $\mathrm{Cl}^{\mathrm{ev}}(n)$ is embedded into $\mathrm{Cl}^+(n) \oplus \mathrm{Cl}^-(n)$ diagonally. Since the two irreducible representations of the algebra $\mathrm{Cl}(n)$ differ from each other by the automorphism χ they become equivalent after restriction to $\mathrm{Cl}^{\mathrm{ev}}(n)$. This proves the first assertion of proposition.

If $n = 4m$ then the restriction of the representation of the algebra $\mathrm{Cl}(n)$ to the even part $\mathrm{Cl}^{\mathrm{ev}}(n)$ decomposes into the direct sum of two non-equivalent representations. Each of these representations in its turn restricts to the group $\mathrm{Spin}(n)$ as an irreducible one because $\mathrm{Spin}(n)$ contains the basis of the algebra $\mathrm{Cl}^{\mathrm{ev}}(n)$ considered as a vector space. □

Definition 62. The *complex spin representation* of the group $\mathrm{Spin}(n)$ is a group homomorphism

$$\Delta_n^c : \mathrm{Spin}(n) \longrightarrow \mathrm{GL}(S, \mathbb{C})$$

obtained by restricting of an irreducible representation $\rho : \mathbb{C}\mathrm{l}(n) \to \mathrm{End}\, S$ of the complexified Clifford algebra $\mathbb{C}\mathrm{l}(n)$ to the group $\mathrm{Spin}(n) \subset \mathbb{C}\mathrm{l}^{\mathrm{ev}}(n) \subset \mathbb{C}\mathrm{l}(n)$.

Proposition 25. *For odd n the representation Δ_n^c is irreducible and both irreducible representations of the algebra $\mathbb{C}l(n)$ give the same representation when restricted to Spin(n).*

For even $n = 2m$ the representation Δ_{2m}^c admits the decomposition

$$\Delta_{2m}^c = (\Delta_{2m}^c)^+ \oplus (\Delta_{2m}^c)^-$$

into the direct sum of non-equivalent irreducible representations of the group Spin(n)*.*

The proof is analogous to the proof of the previous proposition.

Examples of spin representations

1. The representation Δ_3 has dimension 2 yielding a homomorphism (not depending on the choice of an orthonormal basis)

$$\mathrm{Spin}(3) \longrightarrow \mathrm{SU}(2)$$

which is an isomorphism by the dimension counting, i.e. $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

2. Both representations $\Delta_4^{\pm}$ are 2-dimensional. As in previous case, the representation $\Delta_4^+ \oplus \Delta_4^-$ generates an isomorphism

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2).$$

Spin representation in the complex case

Let V be the $(n = 2m)$-dimensional Euclidean vector space identified with the m-dimensional complex vector space. We shall assume that this space is Kähler, i.e. it is provided also with an Hermitian inner product $\langle\cdot,\cdot\rangle$ compatible with the Euclidean one. Denote by V^* the space dual to V with respect to this inner product. Its complexification $V^*_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C}$ may be represented in the form: $V^*_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. So for every vector $v \in V$ the dual covector v^* is decomposed into the sum

$$v^* = v^{1,0} + v^{0,1}.$$

We shall construct a canonical representation of the Clifford algebra $\mathrm{Cl}(n)$ in the space

$$S_{\mathrm{can}} := \Lambda^{0,*}V^* = \bigoplus_{q=0}^{m} \Lambda^{0,q}(V^*).$$

The representation $\rho_{\mathrm{can}} : V \to \mathrm{End}\, S_{\mathrm{can}}$ in this space is determined by the formula

$$\rho_{\mathrm{can}}(v)\eta = v^{0,1} \wedge \eta - v^{0,1} \lrcorner\, \eta$$

where $v \in V, \eta \in \Lambda^{0,q}(V^*)$. It can be checked that

$$\rho_{\mathrm{can}}(v) \circ \rho_{\mathrm{can}}(v)\eta = -\|v\|^2\eta$$

so by universal property the map $\rho_{\mathrm{can}} : V \to \mathrm{End}\, S_{\mathrm{can}}$ extends to a representation of the Clifford algebra

$$\rho_{\mathrm{can}} : \mathrm{Cl}(n) \longrightarrow \mathrm{End}\, S_{\mathrm{can}}.$$

The semispinor spaces coincide in this case with

$$S^+_{\mathrm{can}} = \Lambda^{0,\mathrm{ev}}(V^*), \quad S^-_{\mathrm{can}} = \Lambda^{0,\mathrm{od}}(V^*).$$

3.2 Spinor geometry

3.2.1 Spin structures

Spin structures on vector bundles

Let $p : E \to M$ be a real vector bundle of rank n over a smooth compact oriented Riemannian manifold M. We shall assume that this bundle is *Riemannian*, i.e. in each fibre E_x, $x \in M$, it is given a positive definite inner product $(\cdot,\cdot)$ smoothly depending on the point $x \in M$.

Moreover, we suppose that E is *orientable* , i.e. in each fibre E_x it is given an orientation which depends smoothly on the point $x \in M$. In contrast with the Riemannian structure, existing on any smooth vector bundle, for the orientability of E it is necessary to impose the following topological condition.

Proposition 26. *A bundle $p : E \to M$ is orientable if and only if its 1st Stiefel–Whitney class $w_1(E) = 0$.*

For the proof cf. [7], II.1, Theorem 1.2. Basic properties of characteristic classes may be found in the book [8].

So we assume that $p : E \to M$ is an orientable Riemannian vector bundle of rank n and $P_{\mathrm{SO}}(E)$ is the bundle of oriented orthonormal bases (briefly: frames) in fibres of E. Suppose that $n \geq 2$ so that the homomorphism $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a double covering.

Definition 63. The *spin structure* on the vector bundle $p : E \to M$ is a principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}}(E)$ together with a double covering bundle morphism

$$\Pi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$$

which is Spin-equivariant in the sense that

$$\Pi(sg) = \Pi(s)\pi(g)$$

for any $s \in P_{\mathrm{Spin}}(E), g \in \mathrm{Spin}(n)$. The action of the group $\mathrm{Spin}(n)$ on $P_{\mathrm{SO}}(E)$ is given by the homomorphism $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$.

The given definition may be rewritten in the form of the following commutative diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}}(E) & \xrightarrow{\Pi} & P_{\mathrm{SO}}(E) \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

in which the restriction of Π to the fibers coincides with the homomorphism π.

Proposition 27. *The spin structure on a vector bundle $p : E \to M$ exists if and only if the 2nd Stiefel–Whitney class $w_2(E) = 0$. If this condition is satisfied then different spin structures on E are numerated by the elements of the cohomology group $H^1(M, \mathbb{Z}_2)$.*

The proof of the proposition cf. in [7], II.1, Theorem 1.7.

Definition 64. An oriented Riemannian manifold M is called the *spin manifold* if its tangent bundle TM admits a spin structure.

Examples of spin manifolds

1. A complex manifold M is spin if and only if its 1st Chern class is even, i.e. $c_1(M) \equiv 0 \bmod 2$. This fact follows from the relation $w_2(M) \equiv c_1(M) \bmod 2$.

2. A complex projective space $\mathbb{CP}^n$ is spin if and only if n is odd.

3. Let Σ_g be a compact Riemann surface of genus g. Then on Σ_g there exist 2^{2g} different spin structures.

Spinor and Clifford bundles

Recall, first of all, a general construction of the *adjoint bundle*. Let $\pi : P \to M$ be a principal G-bundle over a manifold M and F is another smooth manifold provided with a smooth action of the group G from the right, i.e. with a homomorphism $\rho : G \to \mathrm{Diff}(F)$. Then we can construct a new bundle $P \times_\rho F$ with fibre F and structure group G.

For that consider the left action of the group G on the product $P \times F$ given by the formula:

$$g(s, f) := (sg^{-1}, \rho(g)f)$$

where $s \in P, f \in F, g \in G$. The quotient $P \times F$ by this action is denoted by $P \times_\rho F$ and consists of the orbits $[s, f]$ of elements $(s, f) \in P \times F$. The projection $\pi_\rho : P \times_\rho F \to M$ is defined by the formula: $\pi_\rho([s, f]) := \pi(s)$.

If $\rho : G \to \mathrm{GL}(V)$ is a linear representation of the group G in a vector space V then the associated bundle $P \times_\rho F$ is a vector bundle over M.

Example 15. Let M be an oriented Riemannian manifold of dimension n and $P = P_{\mathrm{SO}}(M)$ is the principal $\mathrm{SO}(n)$-bundle of frames on M. Denote by ρ the standard action of the group $\mathrm{SO}(n)$ on the space $\mathbb{R}^n$. Then

$$TM = P_{\mathrm{SO}}(M) \times_\rho \mathbb{R}^n.$$

In a more general way, if $E \to M$ is an oriented Riemannian vector bundle over M then

$$E = P_{\mathrm{SO}}(E) \times_\rho \mathbb{R}^n.$$

Return to the Clifford algebras. Every orthogonal transform of the space $V = \mathbb{R}^n$ generates an automorphism of the Clifford algebra $\mathrm{Cl}(n)$. Indeed, such transform maps the tensor algebra $\mathcal{T}(V)$ into itself and preserves the ideal$\mathcal{J}(V)$, determining the Clifford algebra, so it can be pushed down to the Clifford algebra $\mathrm{Cl}(n)$. Thus, we have a representation

$$\mathrm{cl} : \mathrm{SO}(n) \longrightarrow \mathrm{Aut}\,\mathrm{Cl}(n).$$

Definition 65. Let $E \to M$ be an oriented Riemannian vector bundle. Its *Clifford bundle* is a bundle of the form

$$\mathrm{Cl}(E) := P_{\mathrm{SO}}(E) \times_{\mathrm{cl}} \mathrm{Cl}(n).$$

In other words, $\mathrm{Cl}(E)$ is the bundle of Clifford algebras associated with the bundle E of Euclidean vector spaces. Hence, all constructions which we have invented for Clifford algebras, may be extended to the Clifford bundles. In particular, we can introduce on $\mathrm{Cl}(V)$ a natural inner product.

Definition 66. Let E be an oriented Riemannian vector bundle provided with a spin structure $\Pi : P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$, and $\Delta_n : \mathrm{Spin}(n) \to \mathrm{GL}(S)$ is the real spin representation. A *real spinor bundle* over E is the bundle of the form

$$S(E) := P_{\mathrm{Spin}}(E) \times_{\Delta_n} S.$$

If $\Delta_n^c : \mathrm{Spin}(n) \to \mathrm{GL}(S^c, \mathbb{C})$ is the complex spin representation then the bundle

$$S^c(E) := P_{\mathrm{Spin}}(E) \times_{\Delta_n^c} S^c$$

is called the *complex spinor bundle*.

Further on we shall assume that S and S^c are provided with Hermitian structures.

Having the definition of spinor bundles, we can introduce the semispinor bundles. For that consider the section ω^c of the bundle $\mathbb{C}\mathrm{l}(E)$ of rank $n = 2m$ given at the point $x \in M$ by the formula

$$\omega^c = i^m e_1 \cdot \ldots \cdot e_{2m}$$

for an oriented orthonormal basis $\{e_j\}$ of the space E_x. Then $(\omega^c)^2 = 1$ and we can define the *semispinor bundles* $S^c_\pm(E)$ as the (± 1)-eigenbundles of the operator of Clifford multiplication by ω^c. Thus,

$$S^c_\pm(E) = P_{\mathrm{Spin}}(E) \times_{(\Delta^c_{2m})^\pm} S^c.$$

For $n = 4m$ an analogous construction goes through in the real case. Namely, let ω be the section of the bundle $\mathrm{Cl}(E)$ given by the formula

$$\omega = e_1 \cdot \ldots \cdot e_{4m}$$

in terms of an orthonormal basis $\{e_j\}$ of the space E_x. Then $\omega^2 = 1$ and we have the decomposition

$$S(E) = S_+(E) \oplus S_-(E)$$

into the direct sum of (± 1)-eigenbundles of the operator of Clifford multiplication by ω.

3.2.2 Spinor connections

In this section we give a short exposition of the theory of connections. A more detailed presentation and the proofs of formulated results may be found in the book [7].

Connections in principal bundles

Let $\pi : P \to M$ be a principal G-bundle over a smooth manifold M of dimension n. Denote by V the distribution in the tangent bundle TP formed by the tangent spaces V_p of the fibers of the bundle P at points $p \in P$: $V_p = \{v \in P_p : \ \pi_*(v) = 0\}$. The subspace V_p is called the *vertical subspace* at p.

Note that for any principal bundle $\pi : P \to M$ there exists a homomorphism of the Lie algebra $\mathfrak{g}$ of the Lie group G into the Lie algebra $\mathrm{Vect}(P)$ of smooth vector fields on P generated by the right action of the group G on P. Namely, we associate with an element ξ of the Lie algebra $\mathfrak{g}$ the vector field X given at point $p \in P$ by the formula

$$X_p = p_*(\xi) := \frac{d}{dt}\left(p \cdot \exp(t\xi)\right)|_{t=0}.$$

The vector fields X, constructed in this way, are vertical, moreover, for any $v \in V_p$ there exists a vector field X of the described type such that $X_p = v$. In other words, the assignment $\mathfrak{g} \ni \xi \mapsto X_p \in V_p$ allows to identify the vertical subspace V_p with the Lie algebra $\mathfrak{g}$.

Definition 67. A *connection* in a principal bundle $\pi : P \to M$ is a smooth distribution $H : P \ni p \to H_p$ of subspaces $H_p \subset T_pP$, called *horizontal*, having the following properties:

1. the tangent map $\pi_* : H_p \to T_{\pi(p)}M$ is an isomorphism of vector spaces for any $p \in P$;

2. the distribution H is G-invariant, i.e. $g_*H_p = H_{pg}$ for any $p \in P$, $g \in G$ where we denote by g_* the map tangent to the right action of g on P.

In other words, the tangent space T_pP at any point $p \in P$ admits a decomposition $T_pP = V_p \oplus H_p$ into the direct sum of vertical and horizontal subspaces, i.e. a connection is a G-invariant method of choosing the distribution H supplement to the vertical distribution V.

At any point $p \in P$ the horizontal subspace H_p determines the projection $T_pP \to V_p$ parallel to H_p. Using the isomorphism $V_p \cong \mathfrak{g}$, defined above, we can construct a linear map $\omega_p : T_pP \to \mathfrak{g}$. It determines a 1-form ω on P with values in the Lie algebra $\mathfrak{g}$.

This 1-form ω is called the *connection form* on P and has the following properties:

1. ω is *vertical*, i.e. it vanishes on horizontal vectors;

2. for any element $\xi \in \mathfrak{g}$ the following equality

$$\omega(X_p) = \xi$$

   holds for any $p \in P$ where $X_p = p_*(\xi)$;

3. the form ω is *equivariant* in the sense that $g_*(\omega) = \mathrm{Ad}_{g^{-1}}\omega$.

From any smooth 1-form ω with these properties we can reconstruct the connection H by setting:

$$H_p := \mathrm{Ker}\,\omega_p\,.$$

Definition 68. The *curvature* of a connection H in a principal bundle $\pi : P \to M$ is a smooth 2-form Ω on P with values in the Lie algebra $\mathfrak{g}$ defined by the equality:

$$\Omega = d\omega + [\omega, \omega].$$

This form is horizontal, i.e. it vanishes on any pair of vectors if at least one of them is vertical, and equivariant, i.e. it transforms under the action of elements $g \in G$ by the formula:

$$g_*\Omega = \mathrm{Ad}_{g^{-1}}\Omega.$$

Riemannian connections

Let $P_{\mathrm{SO}}(E)$ be the principal $\mathrm{SO}(n)$-bundle of frames (oriented orthonormal bases) of an oriented vector bundle E of rank n over an oriented Riemannian manifold M. The Lie algebra $\mathrm{so}(n)$ of the Lie group $\mathrm{SO}(n)$ coincides with the space of real skew-symmetric $(n \times n)$-matrices so the connection form ω on this bundle may be considered as an $(n \times n)$-matrix $\omega = (\omega_{ij})$, composed of 1-forms ω_{ij} satisfying the relation: $\omega_{ij} + \omega_{ji} = 0$.

The curvature Ω in this case will be given by a matrix $\Omega = (\Omega_{ij})$ composed of 2-forms

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj},$$

and the map Ad acts by the formula: $\mathrm{Ad}_g \omega = g\omega g^{-1}$.

Definition 69. The *covariant derivative* in E is a linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for any $f \in C^\infty(M)$, $s \in \Gamma(E)$.

If X is a smooth tangent vector field on M then the pairing with X generates a linear map

$$\nabla_X : \Gamma(E) \longrightarrow \Gamma(E)$$

called the *covariant derivative along* X.

Let ω be a connection form on $P_{\mathrm{SO}}(E)$ and $e = \{e_1, \ldots, e_n\}$ is a local orthonormal basis of sections of the bundle E in a neighborhood U of a point $x^0 \in M$. Then e determines a local section

$$e : U \longrightarrow P_{\mathrm{SO}}(E)$$

of the bundle $P_{\mathrm{SO}}(E)$ over U. The dual tangent bundle

$$e^* : T^*(P_{\mathrm{SO}}(E)) \longrightarrow T^*U$$

allows to transport the connection form $\omega = (\omega_{ij})$ to U by setting:

$$\tilde{\omega} := e^*\omega.$$

With this remark we can formulate the following

Proposition 28. *Let ω be a connection form on the bundle $P_{SO}(E)$. Then it determines uniquely a covariant derivative ∇ on the bundle E given in terms of a local orthonormal basis of sections of the bundle E by the formula*

$$\nabla e_i = \sum_{j=1}^{n} \tilde{\omega}_{ij} \otimes e_j \tag{3.1}$$

in which $\tilde{\omega} = e^*\omega$.

This covariant derivative is compatible with Riemannian structure on E in the sense that

$$X\langle s, s'\rangle = \langle \nabla_X s, s'\rangle + \langle s, \nabla_X s'\rangle$$

for any smooth tangent vector fields X and any smooth sections $s, s' \in \Gamma(E)$ where $\langle \cdot\,, \cdot\rangle$ denotes the inner product on E.

This covariant derivative is called Riemannian.

Conversely, any covariant derivative ∇ on E, compatible with Riemannian structure, determines a unique connection with connection form ω on $P_{SO}(E)$ determined by Formula (3.1).

The proof cf. in [7], II.4, Proposition 4.4.

Let ∇ be a covariant derivative on the bundle E. Consider the composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\tilde{\nabla}} \Gamma(\Lambda^2(T^*M) \otimes E)$$

where $\tilde{\nabla}$ is a natural extension of the covariant derivative ∇ to the sections of the bundle $T^*M \otimes E$ of the form $\eta \otimes s$ where η is a 1-form on M and s is a section of E. This extension is defined by the formula

$$\tilde{\nabla}(\eta \otimes s) := d\eta \otimes s - \eta \wedge \nabla s.$$

The composition $R := \tilde{\nabla} \circ \nabla$ is called the *Riemannian curvature*.

Proposition 29. *In the notation of Proposition 28 denote by $\Omega = (\Omega_{ij})$ the curvature of the connection form ω. Then in terms of a local orthonormal basis of sections of the bundle E the Riemannian curvature will be given by the formula*

$$Re_i = \sum_{j=1}^{n} \tilde{\Omega}_{ij} \otimes e_j$$

where $\tilde{\Omega} = (\tilde{\Omega}_{ij})$ and $\tilde{\Omega} = e^*\Omega$. For any smooth tangent vector fields X, Y on M we have

$$R_{X,Y}s = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s.$$

The map $R_{X,Y} : \Gamma(E) \to \Gamma(E)$, called the curvature transform, *has the following symmetry property:*

$$\langle R_{X,Y}s, s' \rangle + \langle s, R_{X,Y}s' \rangle = 0.$$

The proof cf. in [7], II.4, Propositions 4.5, 4.6.

Connections in Clifford and spinor bundles

The construction of connections in Clifford bundles is based on the following idea.

Let $\pi : P = P_G \to M$ be a principal G-bundle over a manifold M. Suppose that it is given a faithful representation $\rho : G \to \mathrm{SO}(n)$ of the group G in the space $\mathbb{R}^n$. Denote by $E = E_\rho$ the Riemannian vector bundle over M associated with P. In other words:

$$E = E_\rho = P \times_\rho \mathbb{R}^n.$$

Then for any given connection H in the bundle P we can construct an induced canonical connection H_ρ in the principal bundle

$$P(E) = P(E_\rho) = P \times_\rho \mathrm{SO}(n)$$

where the group G acts from the left on $\mathrm{SO}(n)$ by the homomorphism ρ.

In order to construct the desired connection H_ρ in $P(E)$ we extend the given connection H in the bundle P trivially to the direct product $P \times \mathrm{SO}(n)$ and then push down the obtained connection to $P \times_\rho \mathrm{SO}(n)$ using the invariance of the connection H. This defines the connection H_ρ in the bundle $P(E_\rho)$.

Note that we have a canonical G-equivariant map $i : P \to P(E)$ given by the formula

$$P \ni p \longmapsto [p, \mathrm{Id}]$$

where $[p, h]$ denotes the class of the pair $(p, h) \in P \times \mathrm{SO}(n)$ in the quotient $P \times_\rho \mathrm{SO}(n)$. This map is an embedding due to the faithfulness of the representation ρ.

Proposition 30. *Let ω be the connection form of a connection H in the bundle P_G with curvature Ω. Denote by ω_ρ the form of the induced connection H_ρ in the bundle $P(E_\rho)$ with curvature Ω_ρ. Then*

$$\omega_\rho|_P = \rho_*\omega, \quad \Omega_\rho|_P = \rho_*\Omega$$

where $\rho_* : \mathfrak{g} \to so(n)$ is the Lie algebra homomorphism tangent to the representation $\rho : G \to SO(n)$.

The proof cf. in [7], II.4, Proposition 4.7.

We apply this construction to Clifford bundles. Let $E \to M$ be an oriented Riemannian bundle of rank n provided with a Riemannian connection. Recall that the Clifford bundle $\mathrm{Cl}(E)$ is the bundle associated with the principal $\mathrm{SO}(n)$-bundle $P_{\mathrm{SO}}(E)$ by the homomorphism $\mathrm{cl} : \mathrm{SO}(n) \to \mathrm{Aut}(\mathrm{Cl}(n))$, i.e.

$$\mathrm{Cl}(E) = P_{\mathrm{SO}}(E) \times_{\mathrm{cl}} \mathrm{Cl}(n).$$

Hence, in accordance with the above construction, the given Riemannian connection in E generates in a canonical way a connection in the Clifford bundle $\mathrm{Cl}(E)$. The covariant derivative, corresponding to this connection, has the following characteristic property.

Proposition 31. *The covariant derivative ∇ of the constructed connection in the Clifford bundle $Cl(E)$ acts by the derivation of sections of the Clifford bundle, i.e.*

$$\nabla(\sigma \cdot \tau) = (\nabla\sigma) \cdot \tau + \sigma \cdot (\nabla\tau)$$

for any sections $\sigma, \tau \in \Gamma(Cl(E))$.

Note that under the canonical identification of $\mathrm{Cl}(E)$ with the bundle $\Lambda^*(E)$ the covariant derivative ∇ will correspond to the derivation of $\Lambda^*(E)$ preserving the subbundles $\Lambda^k(E)$. It implies, in particular, that ∇ preserves also the subbundles $\mathrm{Cl}^{\mathrm{ev}}(E)$ and $\mathrm{Cl}^{\mathrm{od}}(E)$, and the volume element $\omega = e_1 \cdot \ldots \cdot e_n$ is parallel with respect to ∇, i.e. $\nabla\omega = 0$. It follows that for $n \equiv 3, 4 \bmod 4$ the subbundles $\mathrm{Cl}^{\pm}(E)$ are also preserved by the derivative ∇.

The proof of these assertions and the following proposition may be found in [7], II.4, Propositions 4.8, 4.10.

Proposition 32. *For any pair of tangent vector fields X, Y in a neighborhood of a point $x \in M$ the curvature transform*

$$R_{X,Y} : Cl(E_x) \longrightarrow Cl(E_x)$$

is a derivation, i.e.

$$R_{X,Y}(\sigma \cdot \tau) = R_{X,Y}(\sigma) \cdot \tau + \sigma \cdot R_{X,Y}(\tau)$$

for any sections $\sigma, \tau \in Cl(E_x)$.

This transform preserves also the subspaces $\mathrm{Cl}^{\mathrm{ev}}(E_x)$, $\mathrm{Cl}^{\mathrm{od}}(E_x)$ and $\mathrm{Cl}^{\pm}(E_x)$.

Suppose now that the bundle E is provided with a spin structure, i.e. it is given a bundle epimorphism

$$\Pi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E).$$

Consider the associated spinor bundle

$$S(E) = P_{\mathrm{Spin}}(E) \times_{\Delta_n} S$$

constructed with the help of the spin representation $\Delta_n : \mathrm{Spin}(n) \to S$. Then the connection in the bundle $P_{\mathrm{SO}}(E)$ may be pulled up with the help of the epimorphism Π to a connection in the bundle $P_{\mathrm{Spin}}(E)$. The obtained connection in its turn generates in a canonical way a connection in the spinor bundle $S(E)$. For this connection the analogues of Propositions 31 and 32 hold.

The covariant derivative of the constructed connection and its curvature may be computed explicitly in terms of a local orthonormal basis of sections of the bundle $P_{\mathrm{SO}}(E)$. We shall give here only the formulation of this result referring for its proof to [7], II.4, Theorems 4.14, 4.15.

Proposition 33. *Let ω be a connection form on the bundle $P_{SO}(E)$ and $S(E)$ is the spinor bundle associated with E. Then the covariant derivative of the connection in $S(E)$, constructed from the connection form ω, is given in terms of a local orthonormal basis $e = \{e_i\}$ of sections of the bundle E by the formula*

$$\nabla s_k = \frac{1}{2} \sum_{i<j} \tilde{\omega}_{ij} \otimes e_i e_j \cdot s_k$$

where $s = \{s_k\}$ is a local section of the bundle $P_{SO}(S(E))$, determined by the section e, and $\tilde{\omega} = e^*\omega$.

If Ω is the curvature of the connection ω then the curvature R of the spinor connection ∇ in $S(E)$ is given by the formula

$$Rs = \frac{1}{2} \sum_{i<j} \tilde{\Omega}_{ij} \otimes e_i e_j \cdot s$$

where s is a section of the bundle $S(E)$, and $\tilde{\Omega} = e^*\Omega$. In particular, for any pair of tangent vector fields X, Y in a neighborhood of a point $x \in M$ the curvature transform

$$R_{X,Y} : S(E_x) \longrightarrow S(E_x)$$

is given by the formula

$$R_{X,Y} s = \frac{1}{2} \sum_{i<j} \langle R_{X,Y}(e_i), e_j \rangle e_i e_j \cdot s.$$

In the case when $E = TM$ we denote $P_{\mathrm{SO}}(M) := P_{\mathrm{SO}}(TM)$ and $\mathrm{Cl}(M) := \mathrm{Cl}(TM)$. If ∇ is a connection in the bundle $P_{\mathrm{SO}}(M)$ then we can construct with its help a tensor field T associating with a pair of tangent vector fields X, Y in a neighborhood of a point $x \in M$ the quantity

$$T_{X,Y} := \nabla_X Y - \nabla_Y X - [X, Y]$$

called the *torsion tensor* of the connection ∇. The tensor T determines a 2-form on the manifold M with values in the tangent bundle TM.

Theorem 16. *Let M be a Riemannian manifold and $P_{SO}(M)$ be the bundle of frames on TM. Then there exists a unique connection in $P_{SO}(M)$ with vanishing torsion tensor.*

This connection is called the *canonical Riemannian connection* or the *Levi-Civita connection* on M. It induces a canonical connection in the Clifford bundle $\mathrm{Cl}(M)$. In the case when M admits a spin structure this connection induces also a canonical Riemannian connection in the bundle $P_{\mathrm{Spin}}(M)$ hence, in any spinor bundle associated with $P_{\mathrm{Spin}}(M)$.

The curvature tensor R of the canonical Riemannian connection on M has the following symmetry properties:

$$R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y = 0,$$
$$\langle R_{X,Y}Z, U\rangle = \langle R_{Z,U}X, Y\rangle$$

for any tangent vector fields X, Y, Z, U in a neighborhood of a point $x \in M$.

3.2.3 Dirac operator

Basic definitions

Let M be an oriented Riemannian manifold. Using the Riemannian metric, we can identify the tangent space T_xM with the cotangent space T_x^*M at any point $x \in M$. Denote by $\mathrm{Cl}(M)$ the Clifford bundle over M. Suppose that S is a bundle of Clifford modules over M so that every fibre S_x is a module over the Clifford algebra $\mathrm{Cl}(T_xM)$. We shall assume also that S is Riemannian and is provided with a Riemannian connection ∇. Denote by $\rho : \Gamma(\mathrm{Cl}(M)) \otimes S \to S$ the Clifford multiplication map on S.

Definition 70. The *Dirac operator* in S, generated by the connection ∇, is a linear differential operator $D : \Gamma(S) \to \Gamma(S)$ of the form

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{j\otimes 1} \Gamma(\mathrm{Cl}(M) \otimes S) \xrightarrow{\rho} \Gamma(S)$$

where the embedding j is generated by the identification of $\Lambda(T^*M) = \Lambda(TM)$ with $\mathrm{Cl}(TM)$.

In terms of a local orthonormal basis $\{e_1, \ldots, e_n\}$ of the bundle TM the Dirac operator D is given by the formula

$$Ds = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} s$$

where $s \in \Gamma(S)$. The operator D^2 is called the *Dirac Laplacian*.

Lemma 17. *Let $f \in C^\infty(M)$ be a smooth function on M considered as the multiplication operator, acting on sections from $\Gamma(S)$. Then*

$$[D, f] = \rho(df).$$

Proof. Since the multiplication by f commutes with the Clifford multiplication, we have for any $s \in \Gamma(S)$:

$$[D, f]s = \rho\nabla(fs) - f\rho(\nabla s) = \rho(df \otimes s) = \rho(df)s.$$

□

Recall that the *principal symbol* of a differential operator $\mathcal{D} : \Gamma(E) \to \Gamma(E)$, given in local coordinates x in a neighborhood of a point $x^0 \in M$ by the formula

$$\mathcal{D} = \sum_{|\alpha| \le m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

is the map associating with every $x \in M$ and every covector $\xi = \sum \xi_j dx_j \in T_x^* M \setminus \{0\}$ the linear map $\sigma_\xi(\mathcal{D}) : E_x \to E_x$ of the form

$$\sigma_\xi(\mathcal{D}) = i^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

The operator $\mathcal{D}$ is called *elliptic* if the map $\sigma_\xi(\mathcal{D}) : E_x \to E_x$ is non-degenerate for any $x \in M$ and $\xi \in T_x^* M \setminus \{0\}$.

Lemma 18. *For any $\xi \in T_x^* M \setminus \{0\}$ the principal symbols of the Dirac operator and Dirac Laplacian are equal to*

$$\sigma_\xi(D) = i\xi \quad \text{and} \quad \sigma_\xi(D^2) = \|\xi\|^2$$

where both operators act on S by the Clifford multiplication. In particular, both operators D and D^2 are elliptic.

Proof. Fix a point $x^0 \in M$ and a local basis $\{e_1, \dots, e_n\}$ of the bundle TM in a neighborhood of x^0. Choose the local coordinates in a neighborhood of x^0 so that $x^0 = 0$ and $e_j = \partial/\partial x_j$ at x^0. Then for any local trivialization of the bundle S in a neighborhood of x^0 the covariant derivative ∇_{e_j} in a neighborhood of x^0 will be written in the form

$$\nabla_{e_j} = \partial/\partial x_j + \text{ terms of 0th order},$$

and the Dirac operator will have the form

$$D = \sum_j e_j \partial/\partial x_j + \text{ terms of 0th order}.$$

So for any covector $\xi = \sum \xi_j dx_j \in T_x^* M \setminus \{0\}$ in a neighborhood of x^0 the following relations hold:

$$\sigma_\xi(D) = i\sum_j e_j\xi_j = i\xi,$$
$$\sigma_\xi(D^2) = \sigma_\xi(D)^2 = -\xi \cdot \xi = \|\xi\|^2.$$

□

General properties

We impose now some natural extra conditions on the bundle S of Clifford modules.

First of all, we shall assume that the bundle S is provided with a Riemannian structure compatible with the Clifford multiplication in the sense that the Clifford multiplication by unit vectors from TM is an orthogonal transform of S so that for any $x \in M$ the following relation

$$\langle es_1, es_2 \rangle = \langle s_1, s_2 \rangle \tag{3.2}$$

holds for any unit vector $e \in T_xM$ and any sections $s_1, s_2 \in \Gamma(S)$ at x.

Moreover, we shall suppose that the bundle S is provided with a Clifford connection ∇ satisfying the following Leibniz rule

$$\nabla(\sigma \cdot s) = (\nabla\sigma) \cdot s + \sigma \cdot (\nabla s)$$

where $\sigma \in \Gamma(\mathrm{Cl}(M))$, $s \in \Gamma(S)$, and $\nabla\sigma$ is determined by the action of the canonical Riemannian connection on the Clifford bundle.

The bundles S of Clifford modules, having these properties, we call briefly the *Dirac bundles*.

Introduce on S an *inner product* given by the formula

$$(s_1, s_2) := \int_M \langle s_1, s_2 \rangle \,\mathrm{vol}$$

where $s_1, s_2 \in \Gamma(S)$, vol is the volume form on M.

Proposition 34. *The Dirac operator D on a Dirac bundle S is formally selfadjoint, i.e.*

$$(Ds_1, s_2) = (s_1, Ds_2)$$

for any smooth sections $s_1, s_2 \in C^\infty(S)$ with compact supports on M.

Proof. Fix a point $x^0 \in M$ and a local basis $\{e_1, \dots, e_n\}$ of the bundle TM in a neighborhood of the point x^0 such that $\nabla_{e_i} e_j = 0$ at x^0. (Such basis may be constructed by choosing it first at x^0 and then extending to a neighborhood of x^0 by the parallel transport along the geodesics with initial point at x^0.) Then the computation at x^0 will give:

$$\begin{aligned}
\langle Ds_1, s_2 \rangle = \sum_j \langle e_j \nabla_{e_j} s_1, s_2 \rangle = &\ \text{(Formula (3.2))} \\
- \sum_j \langle \nabla_{e_j} s_1, e_j \cdot s_2 \rangle = &\ \text{(compatibility with Riemannian structure)} \\
- \sum_j \left\{ e_j \langle s_1, e_j \cdot s_2 \rangle - \langle s_1, (\nabla_{e_j} e_j) s_2 \rangle - \langle s_1, e_j \nabla_{e_j} s_2 \rangle \right\} = &\ \\
- \sum_j e_j \langle s_1, e_j \cdot s_2 \rangle + \langle s_1, Ds_2 \rangle. &
\end{aligned}$$

Introduce a vector field X defined by the equality:

$$\langle X, Y \rangle = -\langle s_1, Y s_2 \rangle$$

for an arbitrary tangent vector field Y. In terms of this vector we can rewrite the first term in the last formula in the above chain of equalities in the form

$$-\sum_j e_j\langle s_1, e_j\cdot s_2\rangle = -\sum_j e_j\langle X, e_j\rangle = \text{ (adding the zero term } -\sum_j\langle X, \nabla_{e_j}e_j\rangle)$$
$$\sum_j\left\{e_j\langle X, e_j\rangle - \langle X, \nabla_{e_j}e_j\rangle\right\} = \text{ (compatibility with Riemannian structure)}$$
$$\sum_j\langle\nabla_{e_j}X, e_j\rangle =: \operatorname{div} X.$$

Thus, we have established that

$$\langle Ds_1, s_2\rangle = \operatorname{div} X + \langle s_1, Ds_2\rangle.$$

Due to the compactness of supports of s_1, s_2, the first term on the right hand side will vanish after the integration over M and we obtain the required assertion. □

Remark 15. In the case of a manifold M with boundary ∂M the above argument yields the following *Stokes formula*

$$(Ds_1, s_2) - (s_1, Ds_2) = \int_{\partial M}\langle\nu\cdot s_1, s_2\rangle\,\mathrm{vol}$$

where ν is the exterior normal to ∂M.

Remark 16. It follows from the general theory of elliptic operators D that any weak solution of the equation $Ds = 0$ is in fact C^∞-smooth. If the manifold M is compact then this theory implies also that the space of solutions of the equation $Ds = 0$ is finite-dimensional.

Denote by $L^2(S)$ the space of L^2-sections of the bundle S obtained by the completion of the space $\Gamma_0^\infty(S)$ of smooth sections of S with compact supports with respect to the L^2-norm, introduced above. The Dirac operator D is a symmetric operator on $\Gamma_0^\infty(S)$ so it admits the completion in $L^2(S)$-norm. The obtained operator is an unbounded selfadjoint operator in $L^2(S)$.

Recall that the Dirac operator D has the principal symbol $i\xi$, and its square D^2 has the principal symbol $\|\xi\|^2$ coinciding with the principal symbol of the Laplace–Beltrami operator on M. If we introduce the *spinor Laplacian* Δ^S given by the formula

$$\Delta^S := -\mathrm{Tr}_g(\tilde{\nabla}^S\circ\nabla^S)$$

where ∇^S is a spinor connection on S then this operator will be related to the Dirac Laplacian by the following formula

$$D^2 = \Delta^S + \frac{1}{4}\,\mathrm{scal}_g$$

where scal_g is the scalar curvature of (M, g). This formula, called the *Lichnerowicz formula* (cf. [7], II.8, Theorem 8.8), is a particular case of a general *Weitzenböck formula*.

Classical Dirac operator

Let $M = \mathbb{R}^n$ and $S = \mathbb{R}^n \times S_0$ where S_0 is a Clifford module over $\mathrm{Cl}(n)$. In this case the Dirac operator D is a differential operator with constant coefficients of the form

$$D = \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j}\,,$$

acting on S_0-valued functions defined on $\mathbb{R}^n$. Here, γ_j are the *Dirac matrices*, i.e. linear maps $\gamma_j : S_0 \to S_0$ satisfying the relations

$$\gamma_j\gamma_k + \gamma_k\gamma_j = -2\delta_{jk}$$

for all $j, k = 1, \ldots, n$. These relations imply that $D^2 = \Delta \cdot \mathrm{Id}$ where $\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^n$.

Particular cases

1. For $n = 1$ the Clifford algebra $\mathrm{Cl}(1) = \mathbb{C}$, and the Dirac operator $D = i\frac{\partial}{\partial x_1}$.

2. For $n = 2$ the Clifford algebra $\mathrm{Cl}(2) = \mathbb{H} = \mathbb{C} \oplus \mathbb{C} = \mathrm{Cl}^{\mathrm{ev}}(2) \oplus \mathrm{Cl}^{\mathrm{od}}(2)$ and D permutes $\mathrm{Cl}^{\mathrm{ev}}(2)$ and $\mathrm{Cl}^{\mathrm{od}}(2)$. Introduce on $\mathbb{H}$ the real coordinates by writing quaternions $q \in \mathbb{H}$ in the form $q = x_0 1 + x_1 e_1 + x_2 e_2 + x_3 e_3$. If we identify $\mathrm{Cl}^{\mathrm{ev}}(2)$ and $\mathrm{Cl}^{\mathrm{od}}(2)$ with $\mathbb{C}$ using the maps $u + e_2 e_1 \leftrightarrow u + iv \leftrightarrow ue_1 + ve_2$ then the operator $D = e_1 \partial/\partial x_1 + e_2 \partial/\partial x_2$ will be given by the matrix

$$D = \begin{pmatrix} 0 & -\partial/\partial z \\ \partial/\partial \bar{z} & 0 \end{pmatrix}$$

where $\partial/\partial \bar{z} = \partial/\partial x_1 + i\partial/\partial x_2$. In other words, the restriction of this operator to the space $\mathrm{Cl}^{\mathrm{ev}}(2)$ coincides with Cauchy–Riemann operator.

3. For $n = 3$ the Clifford algebra $\mathrm{Cl}(3) = \mathbb{H} \oplus \mathbb{H}$, $S_0 = \mathbb{H}$. This algebra has two representations in $\mathbb{H}$ acting in the following way. Identify the space $\mathbb{R}^3$ with the space $\mathrm{Im}\,\mathbb{H}$ of imaginary quaternions by introducing in the space $\mathrm{Im}\,\mathbb{H}$ the standard basis formed by the imaginary units i, j, k. Then the action of these representations in $\mathbb{H}$ will be given by the multiplication by basic quaternions from the left or from the right. Choosing the left action we shall see that the Dirac operator coincides with the operator of the form

$$D = i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$

acting on $\mathbb{H}$-valued functions on the space $\mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$.

4. For $n = 4$ the Clifford algebra $\mathrm{Cl}(4) = \mathrm{Mat}_2(\mathbb{H})$, and $S_0 = \mathbb{H} \oplus \mathbb{H} = \mathrm{Cl}^{\mathrm{ev}}(4) \oplus \mathrm{Cl}^{\mathrm{od}}(4)$. As in the case $n = 2$, the operator D permutes $\mathrm{Cl}^{\mathrm{ev}}(4)$ and $\mathrm{Cl}^{\mathrm{od}}(4)$. We identify the space $\mathbb{R}^4$ with $\mathbb{H}$ by choosing the standard basis in $\mathbb{H}$ formed by $1, i, j, k$. Introduce the quaternion analogue of the Cauchy–Riemann operator, acting on functions $\mathbb{H} \to \mathbb{H}$, by the formula:

$$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$

or in terms of Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

by the formula

$$\frac{\partial}{\partial \bar{q}} = \sigma_0\frac{\partial}{\partial x_0} + \sigma_1\frac{\partial}{\partial x_1} + \sigma_2\frac{\partial}{\partial x_2} + \sigma_3\frac{\partial}{\partial x_3}.$$

Then the Dirac operator D, acting on functions with values in $S_0 = \mathbb{H} \oplus \mathbb{H}$, will be given by the matrix of the form

$$D = \begin{pmatrix} 0 & -\partial/\partial q \\ \partial/\partial \bar{q} & 0 \end{pmatrix}.$$

Other examples of Dirac operators

1. Let $S = \mathrm{Cl}(M)$ be the Clifford bundle over TM considered as a bundle of Clifford modules over $\mathrm{Cl}(M)$ provided with the Clifford multiplication from the left. Then the corresponding Dirac operator coincides with the square root of the Hodge Laplacian and is called the *Dirac–Hodge operator*.

2. Let M be a spin manifold and S is a spinor bundle over M provided with a Riemannian connection. The arising Dirac operator is called the *Atiyah–Singer operator* and plays a key role in the Atiyah–Singer index theorem.

3. Using the isomorphism $\mathrm{Cl}(M) \cong \Lambda^*(M) \equiv \Lambda^*(T^*M)$, we can consider on $\Lambda^*(M)$, along with the exterior derivation operator $d : \Lambda^*(M) \to \Lambda^*(M)$, the formally adjoint operator $d^* : \Lambda^*(M) \to \Lambda^*(M)$ given on sections of $\Lambda^p(M)$ by the formula
$$d^* = (-1)^{np+n+1} * d*$$
where $*$ is the Hodge operator defined by the equality: $\mu \wedge *\nu = \langle \mu, \nu \rangle \,\mathrm{vol}$. Then the Dirac operator on $\mathrm{Cl}(M) \cong \Lambda^*(M)$ will coincide with the operator $D = d + d^*$, and the operator D^2 with the Hodge Laplacian $\Delta = dd^* + d^*d$.

3.2.4 Spin^c-structures

Spin^c-structures on principal bundles

Let M be a compact oriented Riemannian manifold of dimension n and $P_{\mathrm{SO}} \to M$ is the principal $\mathrm{SO}(n)$-bundles of frames (orthonormal bases) on M. Then the Spin^c-structure on $P_{\mathrm{SO}} \to M$ is the pull-back of this bundle to a principal $\mathrm{Spin}^c(n)$-bundle over M.

More formally,

Definition 71. The *Spin^c-structure on the principal bundle* $P_{\mathrm{SO}} \to M$ is a principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c} \to M$ together with a $\mathrm{Spin}^c(n)$-equivariant bundle epimorphism
$$\begin{array}{ccc} P_{\mathrm{Spin}^c} & \longrightarrow & P_{\mathrm{SO}} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$
where $\mathrm{Spin}^c(n)$ acts on the bundle $P_{\mathrm{SO}} \to M$ by the homomorphism $\pi : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$.

Associate with the bundle $P_{\mathrm{Spin}^c} \to M$ a principal $\mathrm{U}(1)$-bundle $P_{\mathrm{U}(1)} \to M$ together with a $\mathrm{Spin}^c(n)$-equivariant bundle epimorphism so that the following diagram
$$\begin{array}{ccc} P_{\mathrm{Spin}^c} & \longrightarrow & P_{\mathrm{U}(1)} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$
is commutative where $\mathrm{Spin}^c(n)$ acts on the bundle $P_{\mathrm{U}(1)} \to M$ by the homomorphism $\delta : \mathrm{Spin}^c(n) \to \mathrm{U}(1)$. The complex line bundle $\mathcal{L} \to M$, associated with the principal

bundle $P_{\mathrm{U}(1)} \to M$, is called the *characteristic bundle*, and its Chern class $c_1(\mathcal{L})$ is called the *characteristic class* of the considered Spin^c-structure. In terms of the introduced bundle the Spin^c-structure may be also defined in the following equivalent way.

Definition 72. Let $P_{\mathrm{SO}} \to M$ be the principal $\mathrm{SO}(n)$-bundle of frames on M. Then a Spin^c-structure on $P_{\mathrm{SO}} \to M$ is determined by a principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c} \to M$ and principal $\mathrm{U}(1)$-bundle $P_{\mathrm{U}(1)} \to M$ together with a $\mathrm{Spin}^c(n)$-equivariant bundle epimorphism

$$\begin{array}{ccc} P_{\mathrm{Spin}^c} & \longrightarrow & P_{\mathrm{SO}} \times P_{\mathrm{U}(1)} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

where $\mathrm{Spin}^c(n)$ acts on the bundle $P_{\mathrm{SO}} \times P_{\mathrm{U}(1)}$ by the homomorphism (π, δ).

The given definition admits an extension to arbitrary oriented Riemannian vector bundles $E \to M$ of rank n over a compact oriented Riemannian manifold M associated with the principal bundle $P_{\mathrm{SO}} \to M$, i.e.

$$E = P_{\mathrm{SO}} \times_{\mathrm{SO}(n)} \mathbb{R}^n.$$

Definition 73. The *Spin^c-structure on a bundle $E \to M$* is an extension of its structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}^c(n)$. In other words, the bundle $E \to M$ admits a Spin^c-structure if it is a bundle associated with a principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c} \to M$, i.e. there exists a $\mathrm{Spin}^c(n)$-equivariant bundle epimorphism making the following diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}^c} \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n & \longrightarrow & E \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

commutative where the group $\mathrm{Spin}^c(n)$ acts on $\mathbb{R}^n$ by the homomorphism $\pi : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$.

If, in particular, we take for E the tangent bundle TM of an n-dimensional Riemannian manifold M then a Spin^c-structure on TM is called the *Spin^c-structure on the manifold M*.

Examples of Spin^c-structures

Proposition 35. *A principal $SO(n)$-bundle $P_{SO} \to M$ admits a Spin^c-structure if and only if its 2nd Stiefel–Whitney class $w_2(P_{SO})$ is the mod 2-reduction of some integral class $c \in H^2(M, \mathbb{Z})$, i.e.*

$$w_2(P_{SO}) \equiv c \, mod \, 2.$$

An analogous assertion holds for oriented Riemannian vector bundles $E \to M$ over M.

Recall that a bundle $P_{\mathrm{SO}} \to M$ admits a Spin-structure if and only if its 2nd Stiefel–Whitney class is vanishes: $w_2(P_{\mathrm{SO}}) = 0$. This implies

Example 16. Any principal bundle $P_{\mathrm{SO}} \to M$, provided with a spin structure, has a canonical Spin^c-structure. In this case the principal Spin^c-bundle $P_{\mathrm{Spin}^c} \to M$ is defined as

$$P_{\mathrm{Spin}^c} = P_{\mathrm{Spin}} \times_\delta \mathrm{U}(1)$$

where $\mathrm{U}(1)$ denotes the trivial $\mathrm{U}(1)$-bundle over M, and the group $\mathrm{Spin}^c(n)$ acts on the bundle in the right hand side as $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$.

Example 17. Any complex vector bundle bundle $E \to M$ has a canonical Spin-structure. Indeed, in this case $w_2(E) \equiv c_1(E) \bmod 2$ so the existence of this Spin-structure follows from Proposition 35 (more precisely, its analogue for vector bundles).

Complex spinor bundles

Let M be a Spin^c-manifold of dimension n. The *complex spinor bundle* over M is a complex vector bundle S of the form

$$S = P_{\mathrm{Spin}^c} \times_{\Delta_n^c} S_0$$

where S_0 is a Clifford module, and $\Delta_n^c : \mathrm{Spin}^c(n) \to \mathrm{GL}(S_0, \mathbb{C})$ is the spin representation. The bundle S is called *fundamental* if the representation Δ_n^c is irreducible.

If n is even then there are two irreducible representations of the complexified Clifford algebra which became equivalent after restriction to $\mathrm{Spin}^c(n)$. So on any Spin^c-manifold there exists only one fundamental spinor bundle.

Spin^c-structures on complex manifolds

If a Spin^c-manifold M is complex than it has a canonical Spin^c-structure

$$S_{\mathrm{can}} = \Lambda^*_{\mathbb{C}}(TM).$$

The Clifford multiplication $\rho : \mathbb{C}\mathrm{l}(M) \to \mathrm{End}\, S_{\mathrm{can}}$ acts in the following way. Associate with an arbitrary tangent vector v the linear map $\rho(v) : \Lambda^*_{\mathbb{C}}(TM) \to \Lambda^*_{\mathbb{C}}(TM)$ given by the formula:

$$\rho(v)\xi = v \wedge \xi - \xi \lrcorner v^*$$

where v^* is the covector Hermitian dual to v. Then $\rho(v)(\rho(v)\xi) = -\|v\|^2\xi$ so the map ρ extends by universal property to a representation of the whole Clifford algebra $\mathbb{C}\mathrm{l}(M)$.

Other Spin^c-structures on the manifold M provided with the canonical Spin^c-structure, may be constructed by multiplying tensorially the canonical spinor bundle S_{can} by some complex line bundle $L \to M$, i.e. by setting

$$S(L) := S_{\mathrm{can}} \otimes L$$

and

$$P_{\mathrm{Spin}^c}(L) := P_{\mathrm{Spin}} \times_\delta P_{\mathrm{U}(1)}(L)$$

where $P_{\mathrm{U}(1)}(L)$ is the principal $\mathrm{U}(1)$-bundle associated with L, and the action of the group $\mathrm{Spin}^c(n)$ on the bundle in the right hand side is given by the homomorphism $(\pi, \delta) : \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)$.

Hence, there is an action of the group $H^2(M,\mathbb{Z})$, parameterizing the equivalence classes of complex line bundles over M, on the space of Spin^c-structures. The quotient by this action, i.e. the space of different Spin^c-structures on M, is identified with the group $H^1(M,\mathbb{Z}_2)$.

Relation to spin structure

The definition of Spin^c-connections in the bundles with Spin^c-structures is completely analogous to the definition of Spin-connections in spinor bundles. Such connections admit the following description.

Proposition 36. *Let M be a Spin^c-manifold which is also spin. Then from any Spin^c-structure on M, corresponding to a complex line bundle $L \to M$, and arbitrary $U(1)$-connection in the associated principal bundle $P_{U(1)} \to M$, we can construct a canonical connection in $P_{Spin^c} \to M$ which is the pull-up of the connection in $P_{SO} \times P_{U(1)}$ given by the tensor product of the canonical Riemannian connection in P_{SO} and given $U(1)$-connection in $P_{U(1)}$.*

The proof cf. in [7], Proposition D.11.

In fact, the spinority assumption, imposed on M, is superfluous. The given construction extends to the case of general Spin^c-manifolds M if one replaces the U(1)-connection on L by the so called *virtual* connection on the virtual bundle $L^{1/2}$ (cf. [7], pp.396-398).

In conclusion we consider the following question: how to describe the spin structure in terms of Spin^c-structure. Let M be a Spin^c-manifold and S is the corresponding complex spinor bundle provided with Hermitian metric.

Proposition 37. *The manifold M is spin if and only if there exists an anti-linear isometry $C : S \to S$ having the following properties:*

1. $C(sf) = (Cs)\bar{f}$ *for* $s \in \Gamma(S), f \in C^\infty(M)$;
2. $C(\sigma s) = \chi(\bar{\sigma})(Cs)$ *for* $s \in \Gamma(S), \sigma \in \Gamma(\mathbb{C}l(M))$;
3. $\langle Cs, Cs' \rangle = \langle s', s \rangle$ *for* $s, s' \in \Gamma(S)$.

The proof cf. in [3], Theorem 9.6.

Chapter 4

NONCOMMUTATIVE SPINOR GEOMETRY

4.1 Spectral triples

Definition 74. The *spectral triple* for an algebra A is a triple $(A, \mathcal{H}, D)$ consisting of a Hilbert space $\mathcal{H}$, a representation π of the algebra A acting in A by bounded linear operators, and a selfadjoint operator D in $\mathcal{H}$ with compact resolvent, satisfying the following property: the commutator $[D, a]$ of the operator D with any element $a \in A$ (more precisely, with the representation operator $\pi(a)$ defined by this element) is a bounded linear operator in $\mathcal{H}$.

Definition 75. The *real spectral triple* for an algebra A with involution is the spectral triple with an anti-unitary operator C in the space $\mathcal{H}$ for which the map

$$b \longmapsto Cb^*C^{-1}$$

determines an action of the opposite algebra A^o on $\mathcal{H}$ commuting with the action of the algebra A, i.e.

$$[a, Cb^*C^{-1}] = 0 \tag{4.1}$$

for all $a, b \in A$.

Recall that an operator $C : \mathcal{H} \to \mathcal{H}$ is called *anti-unitary* if it defines an anti-linear bijection $\mathcal{H} \to \mathcal{H}$ having the following *anti-isometric property*:

$$(C\xi, C\eta) = (\eta, \xi)$$

for all $\xi, \eta \in \mathcal{H}$ and $C^2 = \pm 1$.

If π is an action of the algebra A on $\mathcal{H}$ then the action π^o of the opposite algebra A^o on $\mathcal{H}$ is given by the formula

$$\pi^o(b) := C\pi(b^*)C^{-1}.$$

So Condition (4.1) means that representations π and π^o commute with each other.

4.2 Definition of the noncommutative spinor geometry

In this section we formulate conditions determining the noncommutative spinor geometry.

Dimension

> There is a nonnegative integer n, called the *dimension of geometry*, for which $D^{-1} \in \mathcal{L}^{n+}(\mathcal{H})$ but $D^{-1} \notin \mathcal{L}_0^{n+}(\mathcal{H})$. It implies, in particular, that the operator $|D|^{-n}$ has finite Dixmier trace not equal to zero.

In the case when the algebra A and the Hilbert space $\mathcal{H}$ are finite-dimensional the dimension of geometry is set to zero.

Note that the dimension of geometry is uniquely defined by the above condition. Indeed, if an operator $T \in \mathcal{L}^p(\mathcal{H})$ with $p \leq n$ then $|T|^n$ is of trace class, hence its Dixmier trace is equal to zero. In particular, if $D^{-1} \in \mathcal{L}^{r+}(\mathcal{H})$ with $r < n$ then $D^{-1} \in \mathcal{L}^p(\mathcal{H})$ with $p = \frac{n+r}{2}$ so $D^{-1} \in \mathcal{L}_0^{n+}(\mathcal{H})$ violating our assumption.

4.2.1 Regularity

> The given spectral triple $(A, \mathcal{H}, D)$ should be *regular*. It means that the algebra $\tilde{A}_D := A \cup [D, A]$, generated by the algebra A and all operators of the form $[D, a]$ with $a \in A$, should satisfy the following condition
>
> $$\tilde{A}_D = A \cup [D, A] \subset \mathrm{Dom}^\infty \delta$$
>
> where $\delta(T) := [|D|, T]$.

The formulated condition means, in other words, that the algebra $\tilde{A}_D$ belongs to the smooth definition domain $\mathrm{Dom}^\infty \delta$ of the derivation operator δ. In particular, the operators of the form $[D, a]$ belong to the definition domains $\mathrm{Dom}\, \delta^k$ of all natural powers δ^k of operator δ.

Let us consider this condition in more detail. For a regular spectral triple $(A, \mathcal{H}, D)$ we can introduce the *Sobolev scale* of Hilbert spaces

$$\mathcal{H}^s := \mathrm{Dom}\, |D|^s, \quad s \in \mathbb{R},$$

with the norm:

$$\|\xi\|_s^2 := \|\xi\|^2 + \| \, |D|^s \xi\|^2.$$

For $s > t$ there is a continuous embedding $\mathcal{H}^s \subset \mathcal{H}^t$ and the intersection

$$\mathcal{H}^\infty := \bigcap_{s \in \mathbb{R}} \mathcal{H}^s = \bigcap_{k=0}^{\infty} \mathcal{H}^k = \mathrm{Dom}^\infty |D|$$

is a Frechet space with the metric given by the family of seminorms $\| \cdot \|_k$, $k \in \mathbb{N}$.

Denote by Op_D^r the space of *operators of rth order*, i.e. linear operators $T : \mathcal{H}^\infty \to \mathcal{H}^\infty$ such that for any $s \in \mathbb{R}$ there exists a positive constant C_s for which

$$\|T\xi\|_{s-r} \leq C_s \|\xi\|_s.$$

In other words, the operator T extends to a bounded operator $\mathcal{H}^s \to \mathcal{H}^{s-r}$.

The regularity condition implies that $\tilde{A}_D \subset \mathrm{Op}^0_D$, and the operators $b - |D|b|D|^{-1} \in \mathrm{Op}^{-1}_D$ for any $b \in \tilde{A}_D$.

4.2.2 Finiteness

The algebra A is a pre-C^*-algebra and the space of smooth vectors

$$\mathcal{H}^\infty = \bigcap_{k=0}^{\infty} \mathrm{Dom}\, D^k$$

is a finitely generated projective A-module.

Starting from the first condition, recall that the *pre-C^*-algebra* is a subalgebra A in a C^*-algebra B which is complete with respect to some locally convex topology, being finer than the topology of B, and close with respect to the holomorphic functional analysis.

In more detail, since the algebra $A \subset \mathrm{Dom}^\infty \delta$ we can introduce on it a Frechet topology generated by the seminorms $a \mapsto \|\delta^k(a)\|$, $k \in \mathbb{N}$. If the algebra A is complete in this topology then

$$A = \bigcap_n A_n$$

where A_n is the Banach algebra obtained by the completion of the algebra A with respect to the norm $a \mapsto \sum_{k=0}^n \|\delta^k(a)\|$. In this case the algebra A evidently satisfies the first condition. Motivated by this observation, we introduce this requirement into the finiteness condition also in the general case.

Consider now the second requirement in the formulation of the finiteness condition. According to it, we can find a number $m \in \mathbb{N}$ and an idempotent $e \in \mathrm{Mat}_m(A)$ for which there is an isomorphism $\mathcal{H}^\infty \to {}^m Ae$ of left A-modules. Replacing the idempotent e by the projector p from the algebra $\mathrm{Mat}_m(A)$ by Kaplansky formula we shall obtain that $\mathcal{H}^\infty = {}^m Ap$. The algebra of endomorphisms $\mathrm{End}_A \mathcal{H}^\infty$ will then identify with the algebra

$$p\mathrm{Mat}_m(A)p = pA^m \otimes_A {}^m Ap.$$

4.2.3 Reality

A key role in the formulation of this condition plays the notion of real spectral data for an algebra A with involution τ which depends crucially on the parity of the number $j \equiv n \bmod 8$.

Definition 76. The *real spectral data* of index $j \in \mathbb{Z}_8$ for an algebra A with involution τ consist for even j of $(A, \mathcal{H}, D, C, \chi)$, and for odd j of $(A, \mathcal{H}, D, C)$ where:

1. $(A, \mathcal{H}, D)$ is a spectral triple for the algebra A.
2. C is an anti-linear isometry of the space $\mathcal{H}$ compatible with involution τ; the latter means that $CaC^{-1} = \tau(a)$ for all $a \in A$.

3. for even j: χ is the grading operator on $\mathcal{H}$ which anti-commutes with D.
4. the operators D, C, χ satisfy the commutation relations of the form

$$C^2 = \pm 1, \quad CD = \pm DC, \quad C\chi = \pm \chi C$$

where the signs depend on the number j and are listed in two following tables:

$j \bmod 8$	0	2	4	6
$C^2 = \pm 1$	+	−	−	+
$CD = \pm DC$	+	+	+	+
$C\chi = \pm \chi C$	+	−	+	−

and

$j \bmod 8$	1	3	5	7
$C^2 = \pm 1$	+	−	−	+
$CD = \pm DC$	−	+	−	+

The choice of the signs depends crucially on the representation theory of real Clifford algebras $\mathrm{Cl}_{p,q}$ associated with non-degenerate quadratic forms of signature (p, q). This theory is out of the scope of our lectures, we refer the interested reader to the detailed discussion of Clifford algebras $\mathrm{Cl}_{p,q}$ and real spectral triples to [3], Secs.5.1,9.5.

Recall that in the Hilbert space $\mathcal{H}$ we have representations π of the algebra A and π^o of the opposite algebra A^o. Consider the representation $\pi \otimes \pi^o$ of the algebra $A \otimes A^o$, being the tensor product of the algebras A and A^o, given by the formula

$$a \otimes b^o \longmapsto aCb^*C^{-1}$$

where $*$ is the involution in the algebra A. We can introduce an involution in $A \otimes A^o$ by the formula

$$\tau(a \otimes b^o) := b^* \otimes (a^*)^o.$$

The element of the algebra $A \otimes A^o$ in the right hand side corresponds to the operator in $\mathcal{H}$ acting by the formula:

$$b^*CaC^{-1} = CaC^{-1}b^* = C(aC^{-1}b^*C)C^{-1}.$$

Since the conjugation operator C satisfies the condition $C^2 = \pm 1$ we have $C^{-1}b^*C = Cb^*C^{-1}$ and the last off-line formula may be rewritten as the relation

$$b^*CaC^{-1} = C(aCb^*C^{-1})C^{-1}$$

which means that the conjugation operator C is compatible with involution τ. So we are now in the situation to which applies Definition 76 of real spectral data for the algebra $A \otimes A^o$.

The conjugation operator C satisfies the commutation relations

$$C^2 = \pm 1, \quad CD = \pm DC, \quad C\chi = \pm \chi C$$

so that $(A, \mathcal{H}, D, C, \chi)$ constitute real spectral data of index $j \equiv n \bmod 8$ for the algebra $A \otimes A^o$ with involution τ.

We shall also call the conjugation operator C the *real structure for the spectral triple* $(A, \mathcal{H}, D)$.

4.2.4 First order

The representation π^o of the algebra A^o commutes not only with representation π of the algebra A but also with all operators of the form $[D, a]$ with $a \in A$, i.e.

$$\big[[D, a], Cb^*C^{-1}\big] = 0$$

for all $a, b \in A$.

This definition is symmetric with respect to A and A^o since the Jacobi identity implies that

$$\big[[D, a], Cb^*C^{-1}\big] + \big[a, [D, Cb^*C^{-1}]\big] = \big[D, [a, Cb^*C^{-1}]\big] = 0$$

whence

$$\big[a, [D, Cb^*C^{-1}]\big] = 0.$$

4.2.5 Orientation

Using the first order condition, we can construct a representation of Hochschild cochains on A with values in the algebra $A \otimes A^o$. Note, first of all, that this algebra is an A-bimodule with a natural bimodule structure given by the relation

$$a'(a \otimes b^o)a'' := a'aa'' \otimes b^o.$$

The mentioned representation is given on the homogeneous Hochschild k-cochains from $C_k(A, A \otimes A^o)$ by the formula

$$\pi_D\left((a \otimes b^o) \otimes a_1 \otimes \ldots \otimes a_k\right) := aCb^*C^{-1}[D, a_1] \ldots [D, a_k].$$

Now we can formulate a condition determining the volume form.

There exists a Hochschild cycle $c \in Z_n(A, A \otimes A^o)$ such that

$$\pi_D(c) = \chi$$

in the case of even dimension n of our geometry. In the odd case this condition reduces to the relation $\pi_D(c) = 1$.

4.2.6 Poincaré duality

This condition is a reformulation of the classical Poincaré duality in terms of K-theory. Namely, using the index map for the operator D, it is possible to construct (cf. [3], p.485) additive pairings on the K-groups:

$$K_i(A) \times K_i(A) \longrightarrow \mathbb{Z}$$

where $i = 0, 1$. Then

the Poincaré duality means that the constructed additive pairings on the groups $K_0(A)$ and $K_1(A)$ are non-degenerate.

4.2.7 Definition of noncommutative spinor geometry

Definition 77. The *noncommutative spinor geometry* is a spectral triple $(A, \mathcal{H}, D)$ satisfying the seven conditions, formulated above.

4.3 Dirac geometry as a noncommutative spinor geometry

Let M be a compact oriented Riemannian manifold provided with a Spin^c-structure. In other words, it is given a Riemannian spinor bundle S together with an anti-linear isometry $C : S \to S$.

Definition 78. The *Dirac geometry* on M is a five-tuple $\mathcal{G} = (A, \mathcal{H}, D, C, \chi)$ where $(A, \mathcal{H}, D)$ is a spectral triple with

1. $A = C^\infty(M)$.

2. $\mathcal{H} = L^2(M, S)$ is the Hilbert space of spinors obtained by the completion of the space of smooth sections $\Gamma^\infty(M, S)$ with respect to the norm determined by the inner product
$$(s, t) := \int_M \langle s, t \rangle \,\mathrm{vol}$$
where $\langle \cdot\,, \cdot \rangle$ is the Euclidean inner product on S, vol is the volume form on M.

3. D is the operator on $\mathcal{H}$ obtained by the closure of the Dirac operator $D = \rho \circ \nabla$ on $\Gamma^\infty(M, S)$ given by the composition of the Clifford multiplication ρ and spinor connection $\nabla \equiv \nabla^S$.

4. C is a conjugation operator determining the spin structure on M.

5. $\chi = \rho(\omega)$ is the grading operator if the dimension of M is even, and $\chi = 1$ if the dimension of M is odd.

We are going to show that the Dirac geometry on M is a noncommutative spinor geometry in the sense of Definition 74, i.e. it satisfies the seven conditions listed in Sec.4.2. Since the algebra $A = C^\infty(M)$ is commutative some of these conditions simplify. For instance, the opposite algebra A^o in this case coincides with the original algebra A, and the representation π^o of this algebra coincides with the representation π of the algebra A. The relation $[a, Cb^*C^{-1}] = 0$ transforms into the commutativity condition $[a, b] = 0$ for the algebra A.

The Hochschild cycle $c \in Z_n(A, A \otimes A^o)$, determining the orientation, may be considered in this case as an element of $Z_n(A)$. Indeed, the representation π_D from Sec.4.2.5 by bounded operators, acting in $\mathcal{H}$, reduces in this case to the representation of the group of Hochschild chains $C_k(A)$ in $\mathcal{L}(\mathcal{H})$, given by the formula
$$\pi_D(a_0 \otimes \ldots \otimes a_k) = a_0[D, a_1] \ldots [D, a_k].$$

The kernel of this map contains the subcomplex $D_k(A)$, generated by the chains of the form $a_0 \otimes \ldots \otimes 1 \otimes \ldots \otimes a_k$, for which some of the elements a_i is equal to 1. Pushing down to the quotient

$$\Omega_k(A) = C_k(A)/D_k(A),$$

we can consider π_D as an A-module homomorphism $\pi_D : \Omega_k(A) \to \mathcal{L}(\mathcal{H})$.

The main result of this section is the following

Theorem 17. *The Dirac geometry is a noncommutative spinor geometry.*

In other words, it satisfies the conditions, listed in Sec.4.2. We shall give here an idea of the proof referring for the detailed proof to [3], Theorem 11.1.

4.3.1 Dimension

The dimension of the geometry $\mathcal{G}$ coincides with the dimension of the manifold M. Indeed, the square of the Dirac operator D^2 has the principal symbol

$$\sigma_2(D^2)(x,\xi) = \|\xi\|^2$$

and so coincides with the principal symbol of the Laplace–Beltrami operator Δ on M. Hence

$$\sigma_{-n}(|D|^{-n}) = \left(\|\xi\|^2\right)^{-n/2} \cdot \mathrm{Id} = \sigma_{-n}(\Delta^{-n/2}) \cdot \mathrm{Id}$$

where Id is the identity operator on S.

It implies that the operator $|D|^{-n}$ is a measurable operator of Dixmier class. Indeed, the Wodzicki residue in this case is equal to

$$\mathrm{Res}(f|D|^{-n}) = \mathrm{rank}\, S \cdot \mathrm{Res}(f\Delta^{-n/2})$$

for $f \in C^\infty(M)$. The noncommutative integral is written in the form

$$\int f|D|^{-n} = c_n \mathrm{Tr}^+(f|D|^{-n})$$

where

$$c_n = \frac{n(2\pi)^n}{2^{[n/2]}\Omega_n}\,.$$

In particular,

$$\int |D|^{-n} = c_n \mathrm{Tr}^+(|D|^{-n}) = 1,$$

i.e. the operator $|D|^{-n} \in \mathcal{L}^{1+}(\mathcal{H})$ but does not belong to the space $\mathcal{L}_0^{1+}(\mathcal{H})$.

4.3.2 Regularity

Since $[D, f] = \rho(df)$ by Lemma 17 we have

$$\|[D, f]\| = \|\rho(df)\| = \|df\|,$$

i.e. the operator $[D, f]$ is bounded in $\mathcal{H}$ for any $f \in C^\infty(M)$.

For the proof of regularity of the spectral triple $(A, \mathcal{H}, D)$ we have to check also that the algebra $\tilde{A}_D$, generated by A and $[D, A]$, lies in the smooth definition domain $\mathrm{Dom}^\infty\delta$ where $\delta(T) := [|D|, T]$. We refer for the proof of this fact to [3], p. 489.

4.3.3 Finiteness

The algebra $A = C^\infty(M)$ is closed with respect to the holomorphic functional calculus since a function $f \in C^\infty(M)$ is invertible in this algebra if and only if it has no zeros but in this case the inverse function $1/f$ also belongs to $C^\infty(M)$.

The smooth definition domain of the operator D coincides with $\mathcal{H}^\infty = C^\infty(M)$. The latter space is a finitely generated projective module over $C^\infty(M)$ by the Serre–Swan theorem (more precisely, by its smooth version).

4.3.4 Reality

The check of the fact that the five-tuple $(A, \mathcal{H}, D, C, \chi)$ constitute real spectral data for the algebra $A = C^\infty(M)$ may be found in [3], pp. 406-407.

4.3.5 First order

This condition in the considered case takes the form

$$[[D, f], g] = [df, g] = 0$$

for $f, g \in C^\infty(M)$ and is evidently satisfied in the case of the commutative algebra $A = C^\infty(M)$.

4.3.6 Orientation

The required Hochschild n-cycle $c \in Z_n(A)$ coincides in this case with the volume form vol of the oriented Riemannian manifold M (the proof of this fact is given in [3], pp. 489-490).

4.3.7 Poincaré duality

This condition is satisfied since in the case of the algebra $A = C^\infty(M)$ it is reduced to the usual Poincaré duality between the de Rham homology and cohomology of the manifold M (cf. [3], pp. 490-491).

4.4 Noncommutative spinor geometry over the algebra $A = C^\infty(M)$

We call the noncommutative spinor geometry $\mathcal{G} = (A, \mathcal{H}, D, C, \chi)$ *irreducible* if it cannot be represented as a nontrivial direct sum of two other noncommutative spinor geometries $\mathcal{G}_1 = (A, \mathcal{H}_1, D_1, C_1, \chi_1)$ and $\mathcal{G}_2 = (A, \mathcal{H}_2, D_2, C_2, \chi_2)$, i.e. it does not admit the decomposition of the form

$$\mathcal{G} = (A, \mathcal{H}_1 \oplus \mathcal{H}_2, D_1 \oplus S_2, C_1 \oplus C_2, \chi_1 \oplus \chi_2).$$

The main result of this section is the following

Theorem 18. *Let $\mathcal{G} = (A, \mathcal{H}, D, C, \chi)$ be an irreducible noncommutative spinor geometry of dimension n over the algebra $A = C^\infty(M)$ where M is a compact oriented Riemannian manifold. Then*

1. *there exists a unique Riemannian metric $g = g(D)$ on the manifold M with the distance function*
$$d_g(p, q) = \sup\{|f(p) - f(q)| : \ f \in C^\infty(M), \|[D, f]\| \leq 1\}.$$
2. *M is a spin manifold and the Dirac operator $\mathcal{D}$, corresponding to its spin structure, differs from the the original operator D only by terms of 0th order.*

In other words, the noncommutative spinor geometry over the algebra $A = C^\infty(M)$ is a Dirac geometry.

As in the previous theorem we give here only an idea of the proof of Theorem 18 referring for details to the book [3], Theorem 11.2.

4.4.1 Construction of the volume form

We define first the noncommutative integral using the following assertion.

Proposition 38. *If $\mathcal{G}$ is a noncommutative spinor geometry of dimension n over the algebra $A = C^\infty(M)$ then the operator $f|D|^{-n}$ is measurable for any function $f \in C^\infty(M)$.*

The proof is given in [3], Proposition 11.3.

This proposition means, in other words, that the definition of the noncommutative integral
$$\int f|D|^{-n} = \mathrm{Tr}_\omega(f|D|^{-n})$$
does not depend on the choice of the form ω in the definition of Dixmier trace. It can be shown that (cf. [3], pp. 494-500) the introduced integral is *positively defined* in the sense that $\int f|D|^{-n} > 0$ for any positive element f of the algebra A. In particular, this integral is non-degenerate which allows to introduce its density determining the volume form on M.

4.4.2 Construction of the spin structure and metric

A good candidate for the role of spinor module is the space $\mathcal{H}^\infty$ of smooth vectors with respect to the action of operator D on $\mathcal{H}$. This space by the finiteness property is a finitely generated projective A-module on which the algebra $A = C^\infty(M)$ acts by the multiplication operators. By the Serre–Swan theorem $\mathcal{H}^\infty$ coincides with the module $\Gamma^\infty(M, S)$ of smooth sections of some vector bundle $S \to M$.

Proposition 39. *On the space $\mathcal{H}^\infty$ there exists a unique A-valued pairing $\{\cdot, \cdot\}$ such that*
$$(\varphi, \psi) = \int \{\psi, \varphi\}|D|^{-n}$$
for all $\varphi, \psi \in \mathcal{H}^\infty$.

Proof. The space $\mathcal{H}^\infty$ may be identified with the module ${}^mAp$ for some projector $p \in \mathrm{Mat}_m(A)$. On ${}^mAp$ there is a standard Hermitian pairing

$$\{ap, bp\}' \equiv ap\, b^* = \sum_{j,k} a_j p_{jk} b_k^*$$

so we can introduce on $\mathcal{H}^\infty$ a new inner product by setting

$$(\varphi, \psi)' := \int \{\psi, \varphi\}' |D|^{-n}.$$

This inner product is equivalent but, generally speaking, does not coincide with the original inner product (φ, ψ). However,

$$(\varphi, \psi)' = (\varphi, T\psi), \quad \varphi, \psi \in \mathcal{H}^\infty,$$

for some positive invertible operator $T \in \mathcal{L}(\mathcal{H})$. Then for any $f \in C^\infty(M)$ we shall have

$$(\varphi, Tf\psi) = (\varphi, f\psi)' = \int \{f\psi, \varphi\}' |D|^{-n} =$$
$$\int \{\psi, f^*\varphi\}' |D|^{-n} = (f^*\varphi, \psi)' = (f^*\varphi, T\psi) = (\varphi, fT\psi).$$

In other words, the operator T commutes with the action of the algebra A so we can introduce a new inner product on the A-module $\mathcal{H}^\infty$ by setting

$$\{\psi, \varphi\} := \{T^{-1}\psi, \varphi\}'.$$

This A-valued pairing already satisfies the hypothesis of the proposition.

To prove the uniqueness of the introduced pairing note that the difference of two such pairings yields an A-valued bilinear map equal to zero on all functionals of the form $g \mapsto \int fg|D|^n$ c $f \in C^\infty(M)$. In particular, $\int ff^*|D|^n = 0$ which is possible only if $ff^* = 0$ in A, i.e. $f = 0$. □

We have identified $\mathcal{H}^\infty$ with the module $\Gamma^\infty(M, S)$ of smooth sections of the bundle $S \to M$. The first order condition says that

$$[[D, f], g] = 0$$

for all $f, g \in C^\infty(M)$. Using the regularity property we can show that the operator $[D, f]$ preserves the space $\mathcal{H}^\infty$. Hence the above equality implies that $[D, f]$ is the 0th order operator on the space of sections $\Gamma^\infty(M, S)$, i.e. it belongs to the space $\Gamma^\infty(M, \mathrm{End}\, S)$. In other words, this operator is a matrix-valued multiplication operator on $\mathcal{H}^\infty$.

Then for arbitrary $f, g \in C^\infty(M)$ and $\psi \in \mathcal{H}^\infty$ we have

$$[D, fg]\psi = f[D, g]\psi + [D, f]g\psi = f[D, g]\psi + g[D, f]\psi,$$

i.e.

$$[D, fg] = f[D, g] + g[D, f].$$

It means that D is a matrix-valued differential operator of 1st order acting on smooth sections of the bundle $S \to M$.

The principal symbol of this operator is an operator-valued function $\sigma_1(D)$ defined on the cotangent bundle T^*M. We compute it using the following formula from the theory of differential operators:

$$\sigma_1(D)(x,\xi) = \lim_{t\to\infty} \frac{1}{t} e^{-itf(x)} D e^{itf(x)}$$

fulfilled for any function $f \in C^\infty(M)$ such that $df(x) = \xi$. Using the L'Hopital rule in the limit in the right hand side we can rewrite the latter formula as

$$\sigma_1(D)(x,\xi) = \lim_{t\to\infty} \frac{d}{dt} e^{-itf(x)} D e^{itf(x)}.$$

Note that for any 1-form $\eta \in \Omega^1(M)$ the map

$$x \longmapsto \sigma_1(D)(x, \eta_x)$$

determines a smooth section from the space $\Gamma^\infty(M, \operatorname{End} S)$. Denote by $\rho(\eta)$ the section of the form

$$\rho(\eta)(x) := -i\sigma_1(D)(x, \eta_x). \tag{4.2}$$

Then for $\eta = df$, $\xi = df(x)$ we shall have

$$\rho(df)(x) := -i\sigma_1(D)(x,\xi) = -i \lim_{t\to\infty} \frac{d}{dt} e^{-itf(x)} D e^{itf(x)} = \lim_{t\to\infty} e^{-itf(x)} [D, f] e^{itf(x)} = [D, f](x)$$

where we have used the fact that $[D, f]$ is a multiplication operator in the last equality. Hence,

$$[D, f] = \rho(df).$$

The Formula (4.2) determines the Clifford action of 1-forms from $\Omega^1(M)$ on the space $\mathcal{H}^\infty$. Moreover, $-\rho(\eta)^2$ generates a *non-degenerate metric* associated with the quadratic form on T^*M:

$$g_x(\eta_x, \eta_x) = g^{-1}(\eta, \eta)(x) := -\rho(\eta_x)^2. \tag{4.3}$$

This metric is Riemannian since

$$-\rho(\eta)^2(x) = \sigma_2(D^2)(x, \eta_x)$$

coincides with the principal symbol of a positive definite operator.

By Formula (4.3) the Clifford action ρ extends to the whole Clifford bundle $\mathrm{Cl}(M)$. Thus, we have constructed the *spinor bundle* $S \to M$ with the *Clifford action* $\rho : \mathrm{Cl}(M) \to \operatorname{End} S$, i.e. a Spin^c-structure on M.

We turn now to the *distance function* determined by the introduced metric g. Let $f \in C^\infty(M)$ and d_g is the distance function on M associated with the constructed metric. If $\gamma : [0,1] \to M$ is a piecewise smooth curve, connecting two points p and q, then

$$f(q) - f(p) = f(\gamma(1)) - f(\gamma(0)) = \int_0^1 \frac{d}{dt} f(\gamma(t))dt = \int_0^1 df_{\gamma(t)}(\dot\gamma(t))dt = \int_0^1 g_{\gamma(t)}(\mathrm{grad}_{\gamma(t)} f, \dot\gamma(t))dt.$$

Using the Cauchy inequality in the integrand we get

$$|f(q) - f(p)| \le \int_0^1 |g_{\gamma(t)}(\mathrm{grad}_{\gamma(t)} f, \dot\gamma(t))|dt \le \int_0^1 |\mathrm{grad}_{\gamma(t)} f| \cdot |\dot\gamma(t)|dt \le \|\mathrm{grad} f\|_\infty \int_0^1 |\dot\gamma(t)|dt = \|\mathrm{grad} f\|_\infty \ell(\gamma)$$

where $\ell(\gamma)$ is the length of the curve γ. So for $\|\mathrm{grad} f\|_\infty \le 1$

$$|f(q) - f(p)| \le \ell(\gamma)$$

for any piecewise smooth curve γ connecting the points p and q. Hence,

$$|f(q) - f(p)| \le d_g(p, q) \tag{4.4}$$

and

$$\sup\{|f(q) - f(p)| : \ f \in C^\infty(M) \ \ \|\mathrm{grad} f\|_\infty \le 1\} \le d_g(p, q).$$

In fact, Estimate (4.4) is satisfied for any absolutely continuous functions f for which their gradient is defined almost everywhere as an essentially bounded vector field.

In order to show that the supremum in Formula (4.4) coincides with $d_g(p, q)$ take for f the function

$$f(x) \equiv f_p(x) := d_g(p, x).$$

It is a Lipschitz function with Lipschitz norm equal to 1 (by the triangle inequality), and the supremum in Formula (4.4), equal to $d_g(p, q)$, is attained for f.

Taking this into account, we obtain

Proposition 40. *The distance between the points p and q on the manifold M may be computed by the formula*

$$d_g(p, q) = \sup\{|f(q) - f(p)| : \ f \in C^\infty(M) \ \textit{with} \ \|[D, f]\| \le 1\}.$$

Proof. Since $[D, f] = \rho(df)$ we have

$$\|\rho(df)\|_\infty^2 = \sup_{x\in M} \|\rho(df)(x)\|^2 = \sup_{x\in M} g_x^{-1}(d\bar f(x), df(x)) = \sup_{x\in M} g_x(\mathrm{grad}_x \bar f, \mathrm{grad}_x f) = \|\mathrm{grad} f\|_\infty^2$$

which implies the assertion of the proposition. □

Thus, the *distance on* M is *determined completely in terms of the operator* D.

We turn now to the *spinor structure* on M defined in terms of the Spin^c-structure by the conjugation operator C. For the commutative algebra $A = C^\infty(M)$, due to the coincidence of representations π and π^o, we should have the equality

$$Cf^*C^{-1} = f,$$

i.e. the operator C determines an anti-linear automorphism of the bundle C.

If $\eta \in \Omega^1(M)$ is a real 1-form then the operator C intertwines $\rho(\eta)$ with $-\rho(\eta)$. Moreover, C is an anti-untary operator with respect to the pairing $\{\cdot,\cdot\}$ (cf. [3],p. 505). Thus, this operator satisfies the properties listed in Proposition 37, and so does define a spinor structure on M.

4.4.3 Dirac operator

The Dirac operator $\mathcal{D}$ for the introduced spinor structure on M differs, generally speaking, from the original operator D but both has the same principal symbol equal to

$$\sigma_1(\mathcal{D})(x,\eta_x) = -i\rho(\eta)(x) = \sigma_1(D)(x,\eta_x).$$

Hence, these operators differ by a term of 0th order given by a matrix-valued multiplication operator acting in the space $\mathcal{H}^\infty$:

$$\mathcal{D} = D - m \tag{4.5}$$

where $m \in \Gamma^\infty(M, \operatorname{End} S)$. The matrix-valued function m has the same properties as the Dirac operator, namely:

$$m^* = m, \quad \chi m = (-1)^n m\chi, \quad CmC^{-1} = \pm m. \tag{4.6}$$

Since the operators D and $\mathcal{D}$ are elliptic the same is true for their powers so we can consider the noncommutative integrals of the form $\int f|\mathcal{D}|^{-n}$ defined in terms of Wodzicki residue by the formula:

$$\int f|\mathcal{D}|^{-n} = c_n \operatorname{Res}(f|\mathcal{D}|^{-n})$$

where $c_n = \frac{1}{2^{[n/2]}\Omega_n}$. The operator $f|\mathcal{D}|^{-n}$ for $f \in C^\infty(M)$ has degree $-n$ and its principal symbol is equal to

$$f(x)\sigma_{-n}(f|\mathcal{D}|^{-n}) = f(x)\sigma_{-n}(\Delta^{-n/2}) \cdot \mathrm{Id}.$$

The Wodzicki density is given by the formula

$$\operatorname{res}_x(f|\mathcal{D}|^{-n}) = c_n' f(x)\sqrt{\det g_x}\, d^n x$$

where $c_n' = 2^{[n/2]}\Omega_n$, and $\nu_g = \sqrt{\det g_x}\, d^n x$ is the density of the Riemannian metric g. So the integral

$$\int f|\mathcal{D}|^{-n} = \int f\nu_g$$

does not depend on the term m of 0th order.

For operators given by Formula (4.5) with the term m satisfying Relations (4.6), we can introduce an action given by the noncommutative integral of the form

$$S(\mathcal{D}) = \int |\mathcal{D}|^{-n+2}.$$

The direct computation of this integral, carried out in [3], pp. 507-512, shows that this action functional (as a function of m) attains its absolute minimum at $m = 0$ and this minimum is equal to

$$S(D) = -\frac{n-2}{24} \int_M \mathrm{scal}_g \nu_g,$$

i.e. coincides with the *Hilbert–Einstein action.*

Bibliography

[1] H.Bass, *Algebraic K-theory*, Benjamin, New York, 1968.

[2] A.Connes, *Noncommutative Geometry*, Academic Press, London–San Diego, 1994.

[3] J.M.Gracia-Bondia, J.C.Varilly, H.Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, Boston–Basel–Berlin, 2001.

[4] L.Hörmander, *The Analysis of Linear Partial Differential Operators*, Springer, Berlin–Heidelberg, 2003.

[5] M.Khalkhali, *Basic Noncommutative Geometry*, European Mathematical Society, Zürich, 2013.

[6] G.Landi, *An Introduction to Noncommutative Spaces and their Geometries*, Springer, Berlin, 1997.

[7] H.Lawson, M.-L.Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, New Jersey, 1989.

[8] J.Milnor, J.D.Stacheff, *Characteristic Classes*, Princeton University Press, Princeton, 1974.

[9] W.Rudin, *Functional Analysis*, McGraw–Hill, New York, 1991.

[10] A.G.Sergeev, *Lectures on Functional Analysis*, Steklov Institute, Moscow, 2014 (in Russian).

[11] M.E.Taylor, *Pseudodifferential operators*, Springer, New York, 1996.

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